이학박사 학위논문

The Lind Zeta Function and Williams' Decomposition Theorem for Sofic Shift-Reversal Systems of Finite Order

(유한차 역행 소픽 기호 역학계에 대한 린드 제타함수와 윌리엄스 분해 정리)

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The Lind Zeta Function and Williams' Decomposition Theorem for Sofic Shift-Reversal Systems of Finite Order

A dissertation
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of the requirements for the degree of
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by

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Abstract

The Lind Zeta Function and Williams' Decomposition Theorem for Sofic Shift-Reversal Systems of Finite Order

We establish the Lind zeta function for automorphisms of shift spaces of finite order and introduce the generating function of shift-flip systems. A decomposition theorem for the Lind zeta function for reversal systems of finite order is established: we express it in terms of the Lind zeta function for automorphism and the generating functions of flip systems. In the sofic case, the Lind zeta function for reversal systems of finite order can be expressed in terms of matrices.

An analogue of Williams' decomposition theorem for reversal systems of finite order is established. To this end, we introduce the concept of half elementary conjugacy. Every conjugacy between two sofic shift-reversal systems of finite order can be decomposed into the composition of an even number of such half elementary conjugacies.

Keywords: Reversal maps, Reversal systems, Lind zeta functions, Williams' decomposition theorem

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Contents

Introduction

Recently, the concept of time reversibility has been studied in many fields: reversible computing in computer science, reversible reaction in chemistry, reversible process in engineering, reversible process in thermodynamics and quantum mechanics. For instance, in celestial mechanics, time reversibility has been studied in order to find the initial condition and the history of the universe. In thermodynamics, time reversibility has been studied for energy efficiency with the concept of entropy. Using the application of entropy, time reversibility has been studied in information theory. In field of mathematics, time reversibility has been studied in dynamical systems [**17**], ergodic theory [**7**], and automaton theory [**8**]. In symbolic dynamics, the concept of reversibility was introduced in [**10, 14**]. In this dissertation, we deal with reversal for shift spaces. We begin with the definition of reversal systems.

Let (X, T) be an invertible dynamical system. A homeomorphism $R : X \to X$ is said to be a *reversal for* (X, T) if

$$T \circ R = R \circ T^{-1}, \tag{0.1}$$

that is, R is a conjugacy from (X, T) to (X, T^{-1}). The triple (X, T, R) is called a *reversal system*. In particular, if X is a shift space and φ is a reversal for (X, σ_X), then the reversal system (X, σ_X, φ) will be called a *shift-reversal system*; if X is of finite type, it will be called a *shift-reversal system of finite type*, and if X is sofic, it will be called a *sofic shift-reversal system*.

Suppose that (X, T, R) is a reversal system. If $R^m = \mathrm{id}_X$ and $R^k \neq \mathrm{id}_X$ for $k = 1, \cdots, m-1$, then we write $|R| = m$, R is said to be a *reversal of* *order* m, and (X, T, R) is said to be a *reversal system of order* m. If R is of order m for some positive integer m, then R is said to be a *reversal of finite order* and the triple (X, T, R) is said to be a *reversal system of finite order*. In particular, if R is a reversal of order two, then R is said to be a *flip map* (or, simply a *flip*), and the triple (X, T, R) will be called a *flip system*. For instance, if R is of order $2m$ and m is odd, then (X, T, R^m) is a flip system. Similarly, if X is a shift space and φ is a flip for (X, σ_X), then the flip system (X, σ_X, φ) will be called a *shift-flip system*; if X is of finite type, it will be called a *shift-flip system of finite type*, and if X is sofic, it will be called a *sofic shift-flip system*.

From (0.1), it follows that if R is a reversal for (X, T), then R^{2k} is an automorphism of (X, T) and R^{2k-1} is a reversal for (X, T) for all positive integers k. Hence if R is of order m and $T^2 \neq \mathrm{id}_X$, then m must be even.

Two reversal systems (X, T, R) and (X', T', R') are said to be *conjugate* if there is a homeomorphism $\theta : X \to X'$ such that

$$\theta \circ T = T' \circ \theta \qquad \text{and} \qquad \theta \circ R = R' \circ \theta. \tag{0.2}$$

In this case, we write $(X, T, R) \cong (X', T', R')$, and the homeomorphism θ is called a *conjugacy from* (X, T, R) *to* (X', T', R'). For instance, (0.1) implies that if (X, T, R) is a reversal system, then $(X, T, T^n \circ R)$ is a reversal system for all integers n; and from (0.2) T^n is a conjugacy from (X, T, R) to $(X, T, T^{2n} \circ R)$. Since $\theta \circ R = R' \circ \theta$ implies that $\theta \circ R^k = (R')^k \circ \theta$ for all positive integers k, it is obvious that two conjugate reversal systems have the same order.

The following question is a motivation of this dissertation.

> How can we find out whether or not two sofic shift-reversal systems of finite order are conjugate?

We first consider the case of dynamical systems without reversals, and an overview of this dissertation and a short account of the reason why we only consider reversals of finite order in this dissertation will be given. If n is a positive integer, then the number of periodic points in X of period n is denoted by $p_n(X,T)$:

$$p_n(X,T) = |\{x \in X : T^n(x) = x\}|.$$

The Artin-Mazur zeta function $\zeta_{(X,T)}$ is defined by

$$\zeta_{(X,T)}(t) = \exp\Big(\sum_{n=1}^{\infty} \frac{p_n(X,T)}{n} t^n\Big).$$

It is obvious that if two invertible dynamical systems (X,T) and (X',T') are conjugate, then $p_n(X,T) = p_n(X',T')$ for all integers n. As a consequence, $\zeta_{(X,T)}$ is a conjugacy invariant. In general, however, $\zeta_{(X,T)}(t) = \zeta_{(X',T')}(t)$ does not imply $(X,T) \cong (X',T')$ [**16**].

If $\mathcal{A}$ is a finite set and A is a zero-one $\mathcal{A} \times \mathcal{A}$ matrix, then X_A will denote the topological Markov chain determined by A:

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall i \in \mathbb{Z} \quad A(x_i, x_{i+1}) = 1\}.$$

In this case, we denote the restriction of the shift map of $\mathcal{A}^{\mathbb{Z}}$ to X_A by σ_A. It is well known [**16**] that if (X,σ_X) is a shift of finite type, then there is a zero-one square matrix A such that (X_A, σ_A) is conjugate to (X,σ_X). It is also well known [**16, 22**] that there is an equivalence relation between matrices called strong shift equivalence; and that (X_A, σ_A) and (X_B, σ_B) are conjugate if and only if there is a strong shift equivalence from A to B. The result is an immediate consequence of Williams' decomposition theorem; every conjugacy between two topological Markov chains can be decomposed into the composition of a finite number of elementary conjugacies. Nasu extended in [**19**] the decomposition result to the sofic case and presented a necessary and sufficient condition for two sofic shifts to be conjugate.

In this dissertation we introduce a zeta function for reversal systems of finite order and find its explicit formula; when the underlying dynamical system is a sofic shift, we represent it with a finite number of square matrices whose entries are integers. It will be clear through the definition that the zeta function is a conjugacy invariant. We also obtain a decomposition theorem for conjugacies between shift-reversal systems of finite type, which is analogous to Williams' decomposition theorem, and provides a necessary and sufficient condition for two sofic-reversal systems of finite type to be conjugate. This will be done in terms of matrices. This result is extended to the case of sofic shift-reversal systems.

A reversal φ for shift dynamical system (X, σ_X) will be called a *one-block reversal* if

$$x, x' \in X \quad \text{and} \quad x_0 = x'_0 \quad \Rightarrow \varphi(x)_0 = \varphi(x)'_0.$$

Suppose that (X, σ_X, φ) is a shift-reversal system. In §1.4 we will show that the order of φ is finite if and only if there is a shift-reversal system (Y, σ_Y, ψ) such that it is conjugate to (X, σ_X, φ) and that ψ is a one-block reversal. If φ is a one-block reversal for shift-reversal system of finite type (X, σ_X), then (X, σ_X, φ) will be represented by two zero-one square matrices; very if φ is a one-block reversal for sofic shift-reversal system (X, σ_X), then (X, σ_X, φ) will be represented by two zero-one square matrices and a labeling. On the other hand, if the order of reversal φ for shift of finite type (or sofic shift shift) (X, σ_X) is infinite, then we cannot represent (X, σ_X, φ) in terms of a finite number of matrices (or in terms of a finite number of matrices and a labeling). For this reason we only discuss reversals of finite order in this dissertation. Hence, without mentioning order, we assume that every reversal in this dissertation is of finite order.

In Chapter 1, we introduce the generalization of the Artin-Mazur zeta function, which is called the Lind zeta function. Suppose that G is a group, X is a set, and $\alpha : G \times X \to X$ is a G-action on X. The Lind zeta function [**15**] is given by

$$\zeta_\alpha(t) = \exp\left(\sum_H \frac{p_H(\alpha)}{|G/H|} t^{|G/H|}\right),$$

where the sum is taken over all the finite index subgroups H of G_m, and

$$p_H(\alpha) = |\{x \in X : \forall h \in H \ \alpha_m(h, x) = x\}|.$$

Suppose that G_{2m} is the group generated by two elements a and b having the relation $\langle a, b : ab = ba^{-1}, b^{2m} = 1\rangle$ and that (X, T, R) is a reversal system of order $2m$. Then (X, T, R) can be regarded as the action α_{2m} : $G_{2m} \times X \to X$ defined by

$$\alpha_{2m}(a, x) = T(x) \qquad \text{and} \qquad \alpha_{2m}(b, x) = R(x).$$

We first establish the Lind zeta function for automorphisms of finite order. If φ is an automorphism of (X, T), then (X, T, φ) can be regarded as the action $\beta_m : H_m \times X \to X$, where H_m is the group generated by a and b having the relation $\langle a, b : ab = ba,\, b^m = 1 \rangle$ and β_m is defined by

$$\beta_m(a, x) = x \qquad \text{and} \qquad \beta_m(b, x) = x.$$

In [**10**], the Lind zeta function of α_2 is established. When $m = 2$, the number of fixed points in X fixed by T^N and $T^\delta \circ R$ will be denoted by $p_{N,\delta}$ for all positive integers N and $\delta \in \{0, 1\}$. The Lind zeta function of α_2 is given by

$$\zeta_{\alpha_2}(t) = \sqrt{\zeta_{(X,\sigma_X)}(t)} \exp\left(G(t)\right),$$

where $\zeta_{(X,\sigma_X)}$ is the Artin-Mazur zeta function [**1**] (We will see that it is also the Lind zeta function for automorphism id_X of (X, T) in Chapter 1), and

$$G(t) = \sum_{n=1}^{\infty} \left(p_{2n-1,0} t^{2n-1} + \frac{p_{2n,0} + p_{2n,1}}{2} t^{2n} \right).$$

The function $G(t)$ is called the generating function.

The Lind zeta function for reversal systems (X, T, R) of order $2m$ is given by

$$\zeta_{\alpha_{2m}}(t) = \sqrt{\zeta_{\beta_m}(t^2)} \prod_{\substack{2k-1 \mid m \\ 1 \le 2k-1 \le m}} \exp\left(\frac{G_{4k-2}(t^{2k-1})}{2k-1} \right),$$

where $G_{4k-2}(t)$ is the generating function of subspace of (X, T) equipped with a flip R^{2k-1}.

In Chapter 2, we are interested in conjugacies between sofic shift-reversal systems. We first establish an analogue of Williams' decomposition theorem for shift-reversal systems of finite type. We show that a shift-reversal system of finite type completely determined by a pair of zero-one square matrices, called a reversal pair, up to conjugate, and we introduce a relation between two reversal pairs (A, J) and (B, K), which is called a half elementary equivalence. Suppose that $(\mathsf{X}_A, \sigma_A, \varphi)$ and $(\mathsf{X}_B, \sigma_B, \psi)$ are shift-reversal systems of finite type determined by (A, J) and (B, K), respectively. If there is a half elementary equivalence between (A, J) and (B, K), then there is a natural conjugacy γ, called a half elementary conjugacy, from $(\mathsf{X}_A, \sigma_A, \varphi)$ to $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \psi)$. Every conjugacy between two shift-reversal systems of finite type can be decomposed into the composition of an even number of half elementary conjugacies. We generalize the result to the case of sofic shift-reversal systems.

To deal with sofic shift-reversal systems tractably, we express them in terms of matrices. The matrices are obtained from Krieger's joint state chain of a sofic shift, and it is described in Chapter 3.

CHAPTER 1

The Lind Zeta Function for Sofic Shift-Reversal Systems of Finite Order

Suppose that (X, T) is a topological dynamical system. If n is a positive integer, the number of periodic points in X of period n is denoted by $p_n(X, T)$:

$$p_n(X, T) = |\{x \in X : T^n(x) = x\}|.$$

Suppose that the sequence $\{p_n(X, T)^{1/n}\}$ is bounded. Then the Artin-Mazur zeta function $\zeta_{(X,T)}$, found in [**1**], is defined by

$$\zeta_{(X,T)}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X, T)}{n} t^n\right).$$

Suppose that G is a group and that $\alpha : G \times X \to X$ is a G-action on X. For a finite index subgroup H of G (that is, $|G/H| < \infty$), $p_H(\alpha)$ denotes the number of points in X fixed by all elements of H:

$$p_H(\alpha) = |\{x \in X : \forall h \in H, \alpha(h, x) = x\}|.$$

The Lind zeta function ζ_α of the action α is defined by

$$\zeta_\alpha(t) = \exp\left(\sum_{H} \frac{p_H(\alpha)}{|G/H|} t^{|G/H|}\right),$$

in which the sum is over finite index subgroups H. It is clear that if $\alpha : \mathbb{Z} \times X \to X$ is given by $\alpha(n, x) = T^n(x)$, then the Lind zeta function ζ_α becomes the Artin-mazur zeta function $\zeta_{(X,T)}$. Lind defined this function in [**15**] for the case $G = \mathbb{Z}^d$.

In [**10**], this function is generalized to the case that G is the infinite dihedral group D_∞. Recall that the infinite dihedral group D_∞ is the infinite group generated by two elements a and b having the relation $\langle a, b : ab = ba^{-1},\ b^2 = 1\rangle$. If $\alpha_2 : D_\infty \times X \to X$ is the D_∞-action on X defined by

$$\alpha_2(a, x) = T(x) \qquad \text{and} \qquad \alpha_2(b, x) = F(x),$$

then the flip system (X, T, F) can be regarded as the action α_2.

In a similar manner, if G_{2m} is a group generated by two elements a and b having the relation $\langle a, b : ab = ba^{-1},\ b^{2m} = 1\rangle$, then a reversal system (X, T, R) of order $2m$ can be regarded as the action $\alpha_{2m} : G_{2m} \times X \to X$ defined by

$$\alpha_{2m}(a, x) = T(x) \qquad \text{and} \qquad \alpha_{2m}(b, x) = R(x).$$

In this chapter, we deal with the Lind zeta function for reversal systems. To do this, we need to identify all the finite index subgroups of G_{2m}. It is obvious that if H is a finite index subgroup of G_{2m} and K is a finite index subgroup of H, then K is a finite index subgroup of G_{2m}. Let H_m denote the subgroup $\langle a, b^2\rangle$ generated by a and b^2. Then H_m has index two. It follows that if β_m denotes the restriction of α_{2m} to $H_m \times X$, then the Lind zeta function of the action α_{2m} has $(\zeta_{\beta_m}(t^2))^{1/2}$ as a factor. Similarly, the cyclic subgroup $\langle a\rangle$ generated by a has index $2m$. Therefore, the Lind zeta function of α_{2m} has $(\zeta_{(X,T)}(t^{2m}))^{1/2m}$ as a factor. Moreover, if m is odd, then the subgroup generated by a and b^m has index m and it is isomorphic to the infinite dihedral group. In this case the Lind zeta function of the action α_{2m} has $(\zeta_{\alpha_2}(t^m))^{1/m}$as a factor.

To simplify the process, we first consider in §1.2 the Lind zeta function for automorphisms of finite order. The reason why we only discuss reversals of finite order in this dissertation is described in Introduction. Thus, we only need to consider automorphisms of finite order; and if the order of an automorphism φ of a shift of finite type (or sofic shift) (X, σ_X) is finite, then φ can be recoded into the one-block permutation ψ, and (X, σ_X, ψ) can be represented in terms of matrices (or in terms of matrices and a labeling). For detail, see Lemma 1.2.3. (In Proposition 2.9 of [**4**], it is shown that if the order of an automorphism of shift spaces is finite, then the automorphism can be recoded into the one-block permutation.) In §1.3, the Lind zeta function for flip systems are treated. The generating function of flip systems is introduced and it is shown that the Lind zeta function for flip systems is expressed in terms of the Artin-Mazur zeta function and the generating function. We devote §1.4 to the Lind zeta function for reversal systems. We express it in terms of the Lind zeta function for automorphism and the generating functions of flip systems. In each section, we compute the Lind zeta functions in the sofic case on the basis of some proof of the Artin-Mazur zeta function for sofic shifts, which is described in §1.1.

It is well known that if X is a shift of finite type, then the Artin-mazur function $\zeta_{(X,\sigma_X)}$ is the reciprocal of a polynomial; and that if X is sofic, then $\zeta_{(X,\sigma_X)}$ is a rational function [**16, 18**]. In fact, they are $\mathbb{N}$-rational [**2, 20**]. The Lind zeta functions for shift-reversal systems of finite type and sofic shift-reversal systems are rather complicated. In §1.5, we introduce the notion of $\mathbb{N}$-rationality and we show that the Lind zeta function for sofic shift-reversal systems is the composition of the exponential function and a rational function; but it is neither a $\mathbb{N}$-rational series nor the composition of the exponential function and $\mathbb{N}$-rational series.

1.1. Preliminaries: The Numbers of Periodic Points of Sofic Shifts

Let $\mathcal{A}$ be a finite set equipped with the discrete topology. A *shift space* X over $\mathcal{A}$ is a shift-invariant and closed subspace of $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ in the product topology, where $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by

$$\sigma(x)_i = x_{i+1} \qquad (x \in \mathcal{A}^{\mathbb{Z}};\ i \in \mathbb{Z}).$$

The restriction of σ to X is denoted by σ_X. If X is a shift space over $\mathcal{A}$, then there is a set $\mathcal{F}$ of forbidden blocks of X;

$$\mathcal{F} = \bigcup_{n=1}^{\infty} \{w \in \mathcal{A}^n : w \text{ does not occur in } x \in X\}.$$

$\mathcal{F}$ is called a *forbidden set for* X. We denote the set of all admissible blocks of X whose length is $n \in \mathbb{N}$ by $\mathcal{B}_n(x)$:

$$\mathcal{B}_n(X) = \{w \in \mathcal{A}^n : \exists x \in X \ w \text{ occurs in } x\}.$$

X is called a *shift of finite type* if there is a finite forbidden set $\mathcal{F}$ for X. For instance, the golden mean shift is the set of all bi-infinite sequences over $\{0,1\}$ with no two 1's next to each other, and it has a finite forbidden set $\{11\}$.

If $\mathcal{A}$ is a finite set and A is a zero-one $\mathcal{A} \times \mathcal{A}$ matrix, then X_A will denote the topological Markov chain determined by A:

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall i \in \mathbb{Z} \quad A(x_i, x_{i+1}) = 1\}.$$

In this case, we denote the restriction of the shift map of $\mathcal{A}^{\mathbb{Z}}$ to X_A by σ_A.

It is well known [**16**] that if X is a shift of finite type, then there are a finite set $\mathcal{A}$ and zero-one $\mathcal{A} \times \mathcal{A}$ matrices A such that (X, σ_X) is conjugate to (X_A, σ_A). For instance, if we define $\{0,1\} \times \{0,1\}$ matrix A by

$$A(i,j) = \begin{cases} 1 & \text{if } ij = 0, \\ 0 & \text{if } ij = 1 \end{cases}$$

for $i, j \in \{0,1\}$, then (X_A, σ_A) is conjugate to the golden mean shift.

X is called a sofic shift if there is a shift of finite type X_A and a map(labeling) $\mathcal{L} : \mathcal{B}_1(\mathsf{X}_A) \to \mathcal{B}_1(X)$ such that $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring, where $\mathcal{L}_\infty$ is defined by

$$\mathcal{L}_\infty(x)_i = \mathcal{L}(x_i) \qquad (x \in X;\ i \in \mathbb{Z}).$$

For instance, the shift space X over $\{0,1\}$ having a forbidden set $\mathcal{F} = \{10^{2n+1}1 : n \geq 0\}$ is a proper sofic, that is, it is sofic, but it is not a shift of finite type. Since an even number of 0's occurs between any two 1's in every bi-infinite sequence of X, X is called the even shift.

Suppose that $\mathcal{A}$ is a finite set and A is a zero-one $\mathcal{A} \times \mathcal{A}$ matrix. It is well known [**16**] that the number of periodic points $p_n(\mathsf{X}_A, \sigma_A)$ of period n

is

$$p_n(\mathsf{X}_A, \sigma_A) = \mathrm{tr}(A^n) \qquad (n = 1, 2, \cdots);$$

and that the Artin-Mazur zeta function $\zeta_{(\mathsf{X}_A,\sigma_A)}$ is given by

$$\zeta_{(\mathsf{X}_A,\sigma_A)}(t) = \frac{1}{\det(I - tA)}.$$

The number of periodic points of a sofic shift is also well known [**16, 5, 18**]:

THEOREM 1.1.1. *If X is a sofic shift, then there are square matrices $A_1, A_2, \cdots A_r$ over $\mathbb{Z}$ such that*

$$p_n(X, \sigma_X) = \sum_{j=1}^{r} (-1)^{j+1} \mathrm{tr}(A_j^n) \qquad (n = 1, 2, \cdots).$$

A proof of this theorem is found in §6.4 of [**16**]. With the help of the proof, we will compute the Lind zeta function for sofic shift-reversal systems. In the next two sections, the proof will be modified for the case of automorphisms and the case of flip systems, respectively. So we need to illustrate an outline of the proof, and to do this, we refer to [**13**].

Throughout this section, we assume that (X, σ_X) is a sofic shift, $\mathcal{A}$ is a finite set, $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ is a map (labeling), and A is a $\mathcal{A} \times \mathcal{A}$ zero-one matrix satisfying the following property.

(1) $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring.

(2) $\mathcal{L}_\infty$ is right-resolving.

Recall that a right-resolving for $\mathcal{L}_\infty$ is a pair (a_1a_2, b_1b_2) of blocks in $\mathcal{B}_2(\mathsf{X}_A)$ such that

$$a_1 = b_1 \quad \text{and} \quad \mathcal{L}(a_2) = \mathcal{L}(b_2) \quad \Rightarrow \quad a_2 = b_2.$$

At the end of this section, we will verify that the argument is valid with the assumption "$\mathcal{L}_\infty$ has no graph diamonds" instead of (2). We will see that if $\mathcal{L}$ is right-resolving or has no graph diamonds, then $x \in X$ and $\sigma_X^n x = x$ for $n \in \mathbb{N}$ implies that σ_A^n is a permutation of $\mathcal{L}^{-1}(x) \subset \mathsf{X}_A$.

We denote the cardinality of $\mathcal{A}$ by r, and for each integer i, we denote the projection $\mathsf{X}_A \ni y \mapsto y_i \in \mathcal{A}$ by P_i. The following is trivially proved from (2).

LEMMA 1.1.2. *If x is periodic, then P_i is one-to-one on $\mathcal{L}_\infty^{-1}$ for every i. As a consequence, $1 \le |\mathcal{L}_\infty^{-1}(x)| \le r$.*

Let $F(n)$ be the set of points in X fixed by σ_X^n:

$$F(n) = \{x \in X : \sigma_X^n(x) = x\}.$$

Thus $p_n(X, \sigma_X) = |F(n)|$. Suppose that $x \in F(n)$. By Lemma 1.1.2, we have $1 \le |\mathcal{L}_\infty^{-1}(x)| \le r$. Since $\sigma_X^n(x) = x$, it follows that $\sigma_A^n(\mathcal{L}_\infty^{-1}(x)) = \mathcal{L}_\infty^{-1}(x)$. Thus the restriction of σ_A^n to $\mathcal{L}_\infty^{-1}(x)$ is a permutation of $\mathcal{L}_\infty^{-1}(x)$.

If π is a permutation of a finite set, its sign will be denoted by $\operatorname{sgn}(\pi)$. When $<$ is a linear order of $\mathcal{A}$ and f is a one-to-one function from a subset S of $\mathcal{A}$ into $\mathcal{A}$, we set

$$\operatorname{sgn}(f) = (-1)^{N(<,\, f)},$$

where

$$N(<,\, f) = |\{(a, b) \in S \times S : a < b \quad \text{and} \quad f(b) < f(a)\}|.$$

If $g : f(S) \to \mathcal{A}$ is one-to-one again, then

$$\operatorname{sgn}_<(g \circ f) = \operatorname{sgn}_<(g) \operatorname{sgn}_<(f).$$

We also have $\operatorname{sgn}_<(f) = \operatorname{sgn}(f)$ whenever f is a permutation of a subset of $\mathcal{A}$. From here on, we fix a linear order $<$ of $\mathcal{A}$ and write

$$\operatorname{sgn}(f) = \operatorname{sgn}_<(f)$$

for an arbitrary one-to-one function f from a subset $\mathcal{A}$ into $\mathcal{A}$.

For each positive integer $j \le r$, we set

$$\mathcal{A}_j = \{S \subset \mathcal{A} : |S| = j \quad \text{and} \quad |\mathcal{L}(S)| = 1\};$$

for $S_1, S_2 \in \mathcal{A}_j$, we denote by $F(S_1, S_2)$ the set of one-to-one functions $f : S_1 \to S_2$ such that $A(a, f(a)) = 1$ for all $a \in S_1$.

Define $\mathcal{A}_j \times \mathcal{A}_j$ matrix A_j by

$$A_j(S_1, S_2) = \sum_{f \in F(S_1, S_2)} \operatorname{sgn(f)}.$$

Let n be a positive integer. For $j = 1, 2, \cdots, r$, we denote by $\mathcal{I}_j(n)$ the set of all $(n+1)$-blocks $S_0S_1 \cdots S_n$ over the alphabet $\mathcal{A}_j$ such that $S_0 = S_n$ and $\prod_{i=0}^{n-1} A_j(S_i, S_{i+1}) \neq 0$ so that

$$\operatorname{tr}(A_j^n) = \sum_{S_0S_1\cdots S_n \in \mathcal{I}_j(n)} \prod_{i=0}^{n-1} A_j(S_i, S_{i+1}).$$

Lemma 1.1.3. *A $(n+1)$-block $S_0S_1 \cdots S_n$ belongs to $\mathcal{I}_j(n)$ if and only if $|F(S_j, S_{j+1})| = 1$ for $0 \leq j \leq n-1$ and $S_0 = S_n$.*

Proof. If $S_0S_1 \cdots S_n \in \mathcal{I}_j(n)$, then $|F(S_j, S_{j+1})| \geq 1$ for $0 \leq j \leq n-1$. Since $\mathcal{L}_\infty$ is right-resolving, $|F(S_j, S_{j+1})| = 1$ for $0 \leq j \leq n-1$. The converse is trivially true. $\square$

Let $\mathcal{C}(n;\, x)$ denote the set of all σ_A^n-invariant subsets of $\mathcal{L}_\infty^{-1}(x)$:

$$\mathcal{C}(n;\, x) = \{E \subset \mathcal{L}_\infty^{-1}(x) : \sigma_A^n(E) = E\}.$$

The following lemma is proved in §6.4 of [**16**].

Lemma 1.1.4. *Let π be a permutation of a non-empty finite set S and $\mathcal{C} = \{E \subset S : \pi(E) = E\}$. Then*

$$\sum_{E \in \mathcal{C} \setminus \{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\pi|_E) = 1.$$

With the help of this lemma, we obtain

$$p_n(\sigma_X) = \sum_{x \in F(n)} \sum_{E \in \mathcal{C}(n;\, x) \setminus \{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^n|_E).$$

We set $\mathcal{C}(n) = \cup_{x \in F(n)} \mathcal{C}(n;\, x)$ and

$$\mathcal{C}_j(n) = \{E \in \mathcal{C}(n) : |E| = j\} \qquad (j = 1, 2, \cdots, r).$$

If $x, x' \in F(n)$ and $x \neq x'$, then $\mathcal{C}(n;\, x) \cap \mathcal{C}(n;\, x') = \{\varnothing\}$; hence

$$p_n(\sigma_X) = \sum_{E \in \mathcal{C}(n) \setminus \{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^n|_E).$$

It is obvious that $\mathcal{C}(n) \setminus \{\varnothing\} = \bigcup_{j=1}^{r} \mathcal{C}_j(n)$ and the sets $\mathcal{C}_1(n)$, $\mathcal{C}_2(n)$, $\cdots$, $\mathcal{C}_r(n)$ are mutually disjoint. Hence we obtain

$$p_n(\sigma_X) = \sum_{j=1}^{r} (-1)^{j+1} \sum_{E \in \mathcal{C}_j(n)} \operatorname{sgn}(\sigma_A^n|_E).$$

Now, Theorem 1.1.1 is proved by the following lemma:

LEMMA 1.1.5. *If* $E \in \mathcal{C}_j(n)$ *and*

$$S_i = P_i(E) \qquad (0 \leq i \leq n), \tag{1.1}$$

then $S_0 S_1 \cdots S_n \in \mathcal{I}_j(n), A_j(S_i, S_{i+1}) \in \{-1, 1\}$ *for* $0 \leq i \leq n-1$, *and*

$$\operatorname{sgn}(\sigma_A^n|_E) = \prod_{i=0}^{n-1} A_j(S_i, S_{i+1}).$$

Conversely, if $S_0 S_1 \cdots S_n \in \mathcal{I}_j(n)$, *then there is a unique* $E \in \mathcal{C}_j(n)$ *such that* (1.1) *holds.*

PROOF. Suppose that $E \in \mathcal{C}_j(n)$ and the sets $S_0, S_1, \cdots, S_m$ are given by (1.1). Let $x \in F(n)$ be such that $E \subset \mathcal{L}_\infty^{-1}(x)$. Then we have

$$\mathcal{L}(S_i) = \{\mathcal{L}(y_i) : y \in E\} = \{x_i\} \qquad (i = 0, 1, \cdots, n) \tag{1.2}$$

and Lemma 1.1.2 implies that $P_i|_E : E \to S_i$ is a one-to-one correspondence for all i. In particular,

$$|S_i| = |E| = j \qquad (i = 0, 1, \cdots, n).$$

Thus $S_0, S_1, \cdots, S_n \in \mathcal{A}_j$; and since $\sigma_A^n(E) = E$, we have $S_0 = S_n$.

If we set $f_i = P_{i+1} \circ (P_i|_E)^{-1}$ for $i = 0, 1, \cdots, n-1$, then $f_i \in F(S_i, S_{i+1})$ for all i; hence Lemma 1.1.3 implies that $S_0 S_1 \cdots S_n \in \mathcal{I}_j(n)$. By the same lemma, we also have $F(S_i, S_{i+1}) = \{f_i\}$ for all i, and this implies that $A_j(S_i, S_{i+1}) \in \{-1, 1\}$ for all i, and that

$$\prod_{j=0}^{n-1} A_j(S_i, S_{i+1}) = \prod_{i=0}^{n-1} \operatorname{sgn}(f_i) = \operatorname{sgn}(f_{n-1} \circ \cdots f_1 \circ f_0) = \operatorname{sgn}(P_n \circ (P_0|E)^{-1}).$$

Since $P_n(y) = P_0(\sigma_A^n(y))$ for all $y \in \mathsf{X}_A$, we have

$$\operatorname{sgn}(\sigma_A^n|_E) = \operatorname{sgn}(P_0 \circ \sigma_A^n \circ (P_0|_E)^{-1}) = \operatorname{sgn}(P_n \circ (P_0|_E)^{-1}),$$

and this proves the first assertion.

To prove the second assertion, suppose that $S_0 S_1 \cdots S_n \in \mathcal{I}_j(n)$. By Lemma 1.1.3, there are functions $f_0, f_1, \cdots, f_{n-1}$ such that $F(S_i, S_{i+1}) = \{f_i\}$ for all i. If E denotes the set of all $y \in \mathsf{X}_A$ such that

$$l \in \mathbb{Z} \text{ and } 0 \le i \le n-1 \;\Rightarrow\; y_{ln+i} \in S_i \text{ and } y_{ln+i+1} = f_j(y_{ln+i}),$$

then it is easy to show that $E \in \mathcal{C}_j(n)$ and (1.1) holds.

It remains to prove the uniqueness of E, and to do this, suppose that $x, x' \in F(n)$, $E, E' \in \mathcal{C}_j(n)$, $E \subset \mathcal{L}_\infty^{-1}(x)$, $E' \subset \mathcal{L}_\infty^{-1}(x')$ and that $P_i(E) = P_i(E')$ for $0 \le i \le n-1$. Then (1.2) implies that $x = x'$. Hence we have $E = E'$, by Lemma 1.1.2. This completes the proof. $\square$

Remark. We proved Theorem 1.1.1 with the following assumptions:
(1) $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring.
(2) $\mathcal{L}_\infty$ is right-resolving.
The second assumption can be replaced by the following.
(2′) $\mathcal{L}_\infty$ has no graph diamonds.
Recall that a graph diamond for $\mathcal{L}_\infty$ is a pair $(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n)$ of distinct blocks in $\mathcal{B}_n(\mathsf{X}_A)$ such that $a_1 = b_1$, $a_n = b_n$ and $\mathcal{L}(a_i) = \mathcal{L}_(b_i)$ for all i. It is obvious that Lemma 1.1.2 and Lemma 1.1.3 are valid with the assumption (2′).

The Artin-Mazur zeta function of a sofic shift is well known [**16**]:

Corollary 1.1.6. *If X is a sofic shift, then there are square matrices $A_1, A_2, \cdots A_r$ over $\mathbb{Z}$ such that*

$$\zeta_{(X,\sigma_X)}(t) = \prod_{j=1}^{r} [\det(I - tA_j)]^{(-1)^j}.$$

We observe that the Artin-Mazur zeta function for sofic shifts is a rational function. For the $\mathbb{N}$-rationality of this function, see §1.5.

1.2. Automorphisms of Finite Order

Let φ be an automorphism of a dynamical system (X, T) of order m and let H_m be a group generated by two elements a and c such that

$$ac = ca \qquad \text{and} \qquad c^m = 1.$$

The triple (X, T, φ) can be regarded as H_m-action $\beta_m : H_m \times X \to X$ on X defined by

$$\beta_m(a, x) = T(x) \qquad \text{and} \qquad \beta_m(c, x) = \varphi(x). \qquad (x \in X).$$

In this section we establish the Lind zeta function $\zeta_{\beta_m} = \zeta_{(X,T,\varphi)}$ for automorphism φ. Note that H_m is a multiplicative group which is isomorphic to the additive group $\mathbb{Z} \oplus \mathbb{Z}_m$. Thus if $m = 1$, then $\zeta_{(X,T,\varphi)}$ is equal to the Artin-Mazur zeta function $\zeta_{(X,T)}$. So we assume that $m > 1$. We first consider the case that m is prime.

LEMMA 1.2.1. *If m is prime, then all the finite index subgroups of H_m are as follows:*
(1) $\langle a^n \rangle \qquad (n > 0)$,
(2) $\langle a^n c^l \rangle \qquad (n > 0$ and $0 < l < m)$,
(3) $\langle a^n, c \rangle \qquad (n > 0)$.

PROOF. Suppose that K is a finite index subgroup of H_m. Then K must contain a^i for some integer $i \neq 0$.

First we assume that $c^k \notin K$ for all $k = 1, \cdots m-1$. Then K is generated by $a^n c^l$ for some integers n and l with $n \neq 0$ and $l = 0, 1, \cdots m-1$. Since the inverse $a^{-n}c^{m-l}$ of $a^n c^l$ also generates $\langle a^n c^l \rangle$ for all $n \in \mathbb{Z}$ and $l = 0, 1, \cdots m-1$, it is enough to consider the case that n is a positive integer. Hence we obtain (1) and (2).

Now we assume that $c^k \in K$ for some $k = 1, \cdots m-1$. Because m is prime, c^k generates $\langle c \rangle$. This implies that $\langle a^n c^l, c^k \rangle = \langle a^n, c \rangle$ for all $n \in \mathbb{Z}$ and $l = 0, \cdots, m-1$. $\square$

Remark. The indexes of these subgroups are as follows:
(1) $|H_m/\langle a^n\rangle| = mn \qquad (n > 0)$,
(2) $|H_m/\langle a^n c^l\rangle| = mn \qquad (n > 0 \text{ and } 0 < l < m)$,
(3) $|H_m/\langle a^n,\, c\rangle| = n \qquad (n > 0)$.

To illustrate the Lind zeta function, we indicate some notation. If $0 \leq l < m$, then $p_{n\circ l}(X, T, \varphi)$ will be denoted the number of points in X fixed by all elements of $\langle a^n c^l\rangle$:

$$p_{n\circ l}(X, T, \varphi) = |\{x \in X : T^n \circ \varphi^l(x) = x\}|.$$

If $l = 0$, then $p_{n\circ 0}(X, T, \varphi) = p_n(X, T)$. The number of points in X fixed by all elements of $\langle a^n, c\rangle$ will be denoted by $p_{n*1}(X, T)$:

$$p_{n*1}(X, T, \varphi) = \{x \in X : T^n(x) = \varphi(x) = x\}.$$

For a positive divisor k of m, let $X_{(k)}$ denote the subspace of X whose elements are fixed by φ^k:

$$X_{(k)} = \{x \in X : \varphi^k(x) = x\}.$$

If $k = m$, then $X_{(m)} = X$. We denote the restrictions of T and φ to $X_{(k)}$ by T and φ, unless there is a confusion. Then we have

$$p_n(X_{(1)}, T) = p_{n*1}(X, T, \varphi).$$

Thus we conclude that if φ is an automorphism of (X, T) of prime order m, then the Lind zeta function $\zeta_{(X,T,\varphi)}$ is given by

$$\zeta_{(X,T,\varphi)}(t) = \exp\big(g_m(t)\big)\exp\big(g_1(t)\big),$$

where

$$g_m(t) = \sum_{n=1}^{\infty}\sum_{l=0}^{m-1}\frac{p_{n\circ l}(X, T, \varphi)}{mn}t^{mn},$$

and

$$g_1(t) = \sum_{n=1}^{\infty}\frac{p_{n*1}(X, T, \varphi)}{n}t^n = \sum_{n=1}^{\infty}\frac{p_n\left(X_{(1)}, T\right)}{n}t^n.$$

Now we consider the case that m is an arbitrary positive integer greater than 1.

Lemma 1.2.2. *If m is a positive integer greater than 1, then all the finite index subgroups of H_m are as follows:*
(1) $\langle a^n \rangle \qquad (n > 0)$,
(2) $\langle a^n c^l \rangle \qquad (n > 0 \text{ and } 0 < l < m)$,
(3) $\langle a^n,\, c^k \rangle \qquad (n > 0,\ 0 < k < m \text{ and } k|m)$,
(4) $\langle a^n c^l,\, c^k \rangle \qquad (n > 0,\ 0 < l < k < m \text{ and } k|m)$.

Proof. By the similar arguments in the proof of Lemma 1.2.1, (1) and (2) follow. Suppose that k and l are positive integers such that $0 < k,\ l < m$ and $\gcd(l, m) = k$. Then c^l generates $\langle c^k \rangle$. Therefore it suffices to consider the cases (3) and (4) only when k divides m. Now we consider the case (4). If there is an integer j such that $l = jk$, then $\langle a^n c^l, c^k \rangle$ is also generated by two elements a^n and c^k and this subgroup belongs to the case (3). So we assume that k does not divide l. Since $\langle a^n c^l, c^k \rangle = \langle a^n c^{l-k}, c^k \rangle$, we obtain (4) by the division algorithm. □

Remark. The indexes of these subgroups are as follows:
(1) $|H_m/\langle a^n \rangle| = mn \qquad (n > 0)$,
(2) $|H_m/\langle a^n c^l \rangle| = mn \qquad (n > 0 \text{ and } 0 < l < m)$,
(3) $|H_m/\langle a^n,\, c^k \rangle| = kn \qquad (n > 0,\ 0 < k < m \text{ and } k|m)$,
(4) $|H_m/\langle a^n c^l,\, c^k \rangle| = kn \qquad (n > 0,\ 0 < l < k < m \text{ and } k|m)$.

When k divides m, $\langle a, c^k \rangle$ is a finite index subgroup of H_m. We observe that the subgroups $\langle a^n, c^k \rangle$ and $\langle a^n c^l, c^k \rangle$ ($n > 0$ and $0 < l < k$) of H_m are also finite index subgroups of $\langle a, c^k \rangle$.

As in the case that m is prime, we indicate some notation. If n is a positive integer and k is a positive divisor of m, then $p_{n*k}(X, T, \varphi)$ and $p_{(n\circ l)*k}(X, T, \varphi)$ will be denote the numbers of points in X fixed by all elements of $\langle a^n, c^k \rangle$ and $\langle a^n c^l, c^k \rangle$, respectively:

$$p_{n*k}(X, T, \varphi) = |\{x \in X : T^n(x) = \varphi^k(x) = x\}|,$$

$$p_{(n\circ l)*k}(X, T, \varphi) = |\{x \in X : (T^n \circ \varphi^l)(x) = \varphi^k(x) = x\}| \quad (0 \le l < k).$$

Then we have

$$p_{(n\circ 0)*k}(X,T,\varphi) = p_{n*k}(X,T,\varphi), \quad p_{(n\circ l)*m}(X,T,\varphi) = p_{n\circ l}(X,T,\varphi)$$

$$p_n(X_{(k)},T) = p_{n*k}(X,T,\varphi),$$

and

$$p_{n\circ l}(X_{(k)},T,\varphi) = P_{(n\circ l)*k}(X,T,\varphi) \quad (0 \le l < k).$$

Now, the Lind zeta function $\zeta_{(X,T,\varphi)}(t)$ is given by

$$\zeta_{(X,T,\varphi)}(t) = \prod_{\substack{k|m \\ 1\le k\le m}} \exp\big(g_k(t)\big),$$

where

$$g_k(t) = \sum_{n=1}^{\infty}\sum_{l=0}^{k-1} \frac{p_{(n\circ l)*k}(X,T,\varphi)}{kn} t^{kn} = \sum_{n=1}^{\infty}\sum_{l=0}^{k-1} \frac{p_{n\circ l}(X_{(k)},T,\varphi)}{kn} t^{kn}.$$

In the rest of this section, we will compute the Lind zeta function for automorphisms of sofic shifts. We begin with some basic properties. Let (X,T) and (X',T') be dynamical systems. Suppose that φ and ψ are automorphisms of (X,T) and (X',T'), respectively, and that there is a conjugacy $\theta : (X,T) \to (X',T')$ such that

$$\theta \circ \varphi = \psi \circ \theta. \tag{1.3}$$

If φ is of order m for some positive integer $m > 1$, then the order of ψ is also m by (1.3). If n is a positive integer, k is a positive divisor of m, and $0 \le l < k$, then (1.3) implies that

$$p_{(n\circ l)*k}(X,T,\varphi) = p_{(n\circ l)*k}(X',T',\psi).$$

Hence we obtain

$$\zeta_{(X,T,\varphi)}(t) = \zeta_{(X',T',\psi)}(t).$$

It is well known [**16**] that if X is a shift space, and φ is an automorphism of (X,σ_X), then there is a block map $\Phi : \mathcal{B}_{p+q+1}(X) \to \mathcal{B}_1(X)$ for some nonnegative integers p and q such that

$$\varphi(x)_i = \Phi(x_{[i-p,i+q]}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

If p and q are equal to zero, then φ is said to be a *one-block automorphism.*

LEMMA 1.2.3. *Let φ be an automorphism of (X, σ_X) of order m. Then there are a shift space (Y, σ_Y), a one-block automorphism ψ of (Y, σ_Y) of order m, and a conjugacy $\theta : (X, \sigma_X) \to (Y, \sigma_Y)$ satisfying* (1.3).

PROOF. We define $\mathcal{A}$, $\kappa : \mathcal{A} \to \mathcal{A}$, and $\theta : X \to \mathcal{A}^{\mathbb{Z}}$ by

$$\mathcal{A} = \{(a_0, a_1, \cdots, a_{m-1}) : \exists x \in X,\ a_k = \varphi^k(x)_0 \text{ for } k = 0, \cdots, m-1\},$$

$$\kappa(a_0, a_1, \cdots, a_{m-1}) = (a_1, \cdots, a_{m-1}, a_0) \qquad \big((a_0, a_1, \cdots, a_{m-1}) \in \mathcal{A}\big),$$

and

$$\theta(x)_i = \big(x_i, \varphi(x)_i, \cdots, \varphi^{m-1}(x)_i\big) \qquad (x \in X;\ i \in \mathbb{Z}).$$

Then $\mathcal{A}$ is a finite set, $\kappa^m = \mathrm{id}_{\mathcal{A}}$, θ is one-to-one and continuous, and we have

$$\theta \circ \sigma_X = \sigma \circ \theta \qquad \text{and} \qquad \theta \circ \varphi = \kappa_\infty \circ \theta.$$

If Y denotes $\theta(X)$, and ψ denotes the restriction of κ_∞ to Y, then we have the desired result. □

Prior to the sofic case, we need to compute the Lind zeta function for automorphisms of shifts of finite type.

Suppose that $\mathcal{A}$ is a finite set, m is a positive integer, and A and K are zero-one $\mathcal{A} \times \mathcal{A}$ matrices such that

$$AK = KA, \quad K^m = I, \quad \text{and} \quad K^i \neq I \quad (i = 1, \cdots, m-1). \tag{1.4}$$

Since K is zero-one and $K^m = I$, it follows that there is a unique map $\kappa_K : \mathcal{A} \to \mathcal{A}$ such that $\kappa_K^m = \mathrm{id}_{\mathcal{A}}$, $\kappa_K^i \neq \mathrm{id}_{\mathcal{A}}$ $(0 < i < m)$, and that for $a, b \in \mathcal{A}$ we have $K(a, b) = 1$ if and only if $\kappa_K(a) = b$; and since $AK = KA$, we have

$$A(a, b) = A(\kappa_K(a), \kappa_K(b)) \qquad (a, b \in \mathcal{A}). \tag{1.5}$$

We define $\varphi_K : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\varphi_K(x)_i = \kappa_K(x_i) \qquad (x \in \mathcal{A}^{\mathbb{Z}}; i \in \mathbb{Z}),$$

that is, $\varphi_K = (\kappa_K)_\infty$. Since (1.5) implies that $\varphi_K(\mathsf{X}_A) = \mathsf{X}_A$, the restriction of φ_K to X_A is an automorphism of (X_A, σ_A) of order m. We denote the restriction by $\varphi_{K,A}$.

THEOREM 1.2.4. *Suppose that X is a shift of finite type and φ is an automorphism of (X, σ_X) of order m. Then there is a finite set $\mathcal{A}$ and zero-one $\mathcal{A} \times \mathcal{A}$ matrices satisfying* (1.4) *such that*

$$p_{n \circ l}(X, \sigma_X, \varphi) = \operatorname{tr}(A^n K^l) \qquad (n > 0,\ 0 \le l < m).$$

PROOF. We may assume that φ is a one-block automorphism by Lemma 1.2.3. It is obvious that there are a finite set $\mathcal{A}$, $\mathcal{A} \times \mathcal{A}$ zero-one matrices A and K satisfying (1.4), and a conjugacy $\theta : (X, \sigma_X) \to (\mathsf{X}_A, \sigma_A)$ such that

$$\theta \circ \varphi = \varphi_{K,A} \circ \theta.$$

Hence it is enough to show that

$$p_{n \circ l}(\mathsf{X}_A, \sigma_A, \varphi_{K,A}) = \operatorname{tr}(A^n K^l) \qquad (n > 0,\ 0 \le l < m). \tag{1.6}$$

It is obvious that

$$\sigma_A^n \circ \varphi_{K,A}^l(x) = x \quad \Leftrightarrow \quad x_{jn+i} = \kappa_K^{m-jl}(x_i) \qquad (x \in X; j \in \mathbb{Z}).$$

By (1.5), we have

$$p_{n \circ l}(\mathsf{X}_A, \sigma_A, \varphi_{K,A}) = |\{x_0 \cdots x_{n-1} \in \mathcal{B}_n(\mathsf{X}_A) : A(x_{n-1}, \kappa_K^{m-l}(x_0)) = 1\}|.$$

Since $\kappa_K(a)$ is the unique element of $\mathcal{A}$ satisfying $K(a, \kappa_K(a)) = 1$ for $a \in \mathcal{A}$, we have

$$A(a, \kappa_K^{m-l}(b)) = AK^l(a, b) \qquad (a, b \in \mathcal{A};\ 0 \le l < m).$$

Hence (1.6) is proved. $\square$

Suppose that $\mathcal{A}$ is a finite set and that A, K are zero-one $\mathcal{A} \times \mathcal{A}$ matrices satisfying (1.4). For a positive divisor k of m, let $\mathcal{A}_{(k)}$ denote the set of symbols in $\mathcal{A}$ fixed by κ_K^k:

$$\mathcal{A}_{(k)} = \{a \in \mathcal{A} : \kappa_K^k(a) = a\}, \tag{1.7}$$

and $A_{(k)}$ and $K_{(k)}$ denote the restrictions of A and K to $\mathcal{A}_{(k)}$, respectively. Then we obtain the following corollary.

Corollary 1.2.5. *The Lind zeta function* $\zeta_{(\mathsf{X}_A,\sigma_A,\varphi_{K,A})}$ *is given by*

$$\zeta_{(\mathsf{X}_A,\sigma_A,\varphi_{K,A})}(t) = \prod_{\substack{k|m \\ 1\le k\le m}} \exp\big(g_k(t)\big),$$

where

$$g_k(t) = \sum_{n=1}^{\infty}\sum_{l=0}^{k-1} \frac{\operatorname{tr}\big(A_{(k)}^n K_{(k)}^l\big)}{kn} t^{kn}.$$

Now, we are concerned with the sofic case.

Theorem 1.2.6. *If X is a sofic shift and φ is an automorphism of (X,σ_X) of order m for some positive integer m, then there are square matrices A_j and K_j $(j = 1, 2, \cdots r)$ over $\mathbb{Z}$ such that*

$$p_{n\circ l}(X,\sigma_X,\varphi) = \sum_{j=1}^{r} (-1)^{j+1}\operatorname{tr}(A_j^n K_j^l) \qquad (n > 0,\ 0 \le l < m).$$

By Lemma 1.2.3, we may assume φ is a one-block automorphism. If φ is a one-block automorphism of (X,σ_X), then there is a map $\kappa : \mathcal{B}_1(X) \to \mathcal{B}_1(X)$ such that $\varphi = \kappa_\infty|_X$. Throughout this section, we assume that $\mathcal{A}$ is a finite set, $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ is a map (labeling), and A, K are $\mathcal{A}\times\mathcal{A}$ zero-one matrices having the following properties.

(1) $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring.

(2) $\mathcal{L}_\infty$ is right-resolving.

(3) A and K satisfy (1.4).

(4) $\mathcal{L}\circ\kappa_K = \kappa\circ\mathcal{L}$.

Remarks. As in §1.1, the following can replace (2).

(2′) $\mathcal{L}_\infty$ has no graph diamonds.

Note that (4) implies $\mathcal{L}_\infty\circ\varphi_{K,A} = \varphi\circ\mathcal{L}_\infty$.

As in §1.1, we denote the cardinality of $\mathcal{A}$ by r, the projection $\mathsf{X}_A \ni y \mapsto y_i \in \mathcal{A}$ by P_i for each integer i, and we fix a linear order $<$ of $\mathcal{A}$. For a positive integer $j \le r$, we set

$$\mathcal{A}_j = \{S \subset \mathcal{A} : |S| = j \quad \text{and} \quad |\mathcal{L}(S)| = 1\};$$

for $S_1, S_2 \in \mathcal{A}_j$, we denote by $F(S_1, S_2)$ the set of one-to-one functions $f : S_1 \to S_2$ such that $A(a, f(a)) = 1$ for all $a \in S_1$.

When $S, T \subset \mathcal{A}$ and $f : S \to T$ is a one-to-one correspondence, we denote the map from $\kappa_K(S) \to \kappa_K(T)$ given by

$$\kappa_K(a) \mapsto \kappa_K(f(a)) \qquad (a \in S)$$

by $\kappa_K(f)$. Then $\kappa_K(f) : \kappa_K(S) \to \kappa_K(T)$ is a one-to-one correspondence satisfying $\kappa_K(f) = \kappa_K|_T \circ f \circ \kappa_K^{m-1}|_S$ and

$$\operatorname{sgn}(\kappa_K(f)) = \operatorname{sgn}(\kappa_K|_T)\operatorname{sgn}(f)\operatorname{sgn}(\kappa_K^{m-1}|_S). \tag{1.8}$$

It is clear that $\kappa_K^m(f) = f$. Since $\mathcal{L} \circ \kappa_K = \kappa \circ \mathcal{L}$, we have $\kappa_K(S) \in \mathcal{A}_j$ whenever $S \in \mathcal{A}_j$, and by (1.5), we have $\kappa_K(f) \in F(\kappa_K(S_1), \kappa_K(S_2))$ whenever $f \in F(S_1, S_2)$. It is clear that

$$F(S_1, S_2) \ni f \mapsto \kappa_K(f) \in F(\kappa_K(S_1), \kappa_K(S_2)) \tag{1.9}$$

is a one-to-one correspondence.

Define $\mathcal{A}_j \times \mathcal{A}_j$ matrices A_j and K_j by

$$A_j(S_1, S_2) = \sum_{f \in F(S_1, S_2)} \operatorname{sgn(f)}$$

and

$$K_j(S_1, S_2) = \begin{cases} \operatorname{sgn}(\kappa_K|_{S_1}) & \text{if } \kappa_K(S_1) = S_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that $K_j^m = I$. A simple calculation using (1.8) and the fact that the mapping given by (1.9) is a one-to-one correspondence gives

$$A_j(\kappa_K(S_1), \kappa_K(S_2)) = {K_j}^{m-1}(\kappa_K(S_1), S_1) A_j(S_1, S_2) K_j(S_2, \kappa_K(S_2)), \tag{1.10}$$

for all $S_1, S_2 \in \mathcal{A}_j$, which may be expressed as $A_j = K_j^{m-1} A_j K_j$, or equivalently, as $A_j K_j = K_j A_j$.

Let n be a positive integer and l be a nonnegative integer with $l < m$. We put

$$F(n \circ l) = \{x \in X : \sigma_X^n \circ \varphi^l(x) = x\}$$

so that $p_{n\circ l}(X, \sigma_X, \varphi) = |F(n \circ l)|$. If $x \in F(n \circ l)$, then $\varphi^l(x) = \sigma_X^{-n}(x)$. Therefore x is a periodic point of period mn, and by Lemma 1.1.2, we have $1 \leq |\mathcal{L}_\infty^{-1}(x)| \leq r$. Since $\sigma_X^n \circ \varphi^l(x) = x$, it follows that $\sigma_A^n \circ \varphi_{K,A}^l(\mathcal{L}_\infty^{-1}(x)) = \mathcal{L}_\infty^{-1}(x)$. Thus the restriction of $\sigma_A^n \circ \varphi_{K,A}^l$ to $\mathcal{L}_\infty^{-1}(x)$ is a permutation of $\mathcal{L}_\infty^{-1}(x)$. Let $\mathcal{C}(n \circ l;\, x)$ denote the set of all $(\sigma_A^n \circ \varphi_{K,A}^l)$-invariant subsets of $\mathcal{L}_\infty^{-1}(x)$:

$$\mathcal{C}(n \circ l;\, x) = \{E \subset \mathcal{L}_\infty^{-1}(x) : \sigma_A^n \circ \varphi_{K,A}^l(E) = E\}.$$

Now, we obtain the following from Lemma 1.1.4:

$$p_{n\circ l}(X, \sigma_X, \varphi) = \sum_{x \in F(n\circ l)} \sum_{E \in \mathcal{C}(n\circ l;\, x)\setminus\{\varnothing\}} (-1)^{|E|+1} \mathrm{sgn}(\sigma_A^n \circ \varphi_{K,A}^l|_E).$$

If we set $\mathcal{C}(n \circ l) = \bigcup_{x \in F(n\circ l)} \mathcal{C}(n \circ l;\, x)$, and

$$\mathcal{C}_j(n \circ l) = \{E \subset \mathcal{C}(n \circ l) : |E| = j\} \qquad (j = 1, 2, \cdots, r),$$

then as in the proof of Theorem 1.1.1,

$$p_{n\circ l}(X, \sigma_X, \varphi) = \sum_{j=1}^{r} (-1)^{j+1} \sum_{E \in \mathcal{C}_j(n\circ l)} \mathrm{sgn}(\sigma_A^n \circ \varphi_{K,A}^l|_E).$$

LEMMA 1.2.7. *If* $E \in \mathcal{C}_j(n \circ l)$ *and*

$$S_i = P_i(E) \qquad (0 \leq i \leq n) \tag{1.11}$$

then $A_j(S_i, S_{i+1}) \in \{-1, 1\}$ *for* $0 \leq i \leq n-1$, $S_{jn+i} = \kappa_K^{m-jl}(S_i)$ *for* $0 \leq j < m$, $0 \leq i < n$, $S_{mn} = S_0$, *and*

$$\mathrm{sgn}(\sigma_A^n \circ \varphi_{K,A}^l|_E) = \prod_{i=0}^{n-1} A_j(S_i, S_{i+1}) K_j^l(S_n, S_0).$$

Conversely, if $S_0, \cdots, S_n \in \mathcal{A}_j$, $\prod_{i=0}^{n-1} A_j(S_i, S_{i+1}) \neq 0$, $S_{jn+i} = \kappa_K^{m-jl}(S_i)$ *for* $0 \leq j < m$, $0 \leq i < n$, *and* $S_{mn} = S_0$, *then there is a unique* $E \in \mathcal{C}_j(n \circ l)$ *such that* (1.11) *holds.*

PROOF. Suppose that $E \in \mathcal{C}_j(n \circ l)$ and $S_i = P_i(E)$ for $0 \leq i \leq mn$. Let $x \in F(n \circ l)$ be such that $E \subset \mathcal{L}_\infty^{-1}(x)$. Since x is a periodic point of period mn, $P_i|_E : E \to S_i$ is a one-to-one correspondence for all $i = 0, 1, \cdots, mn$ by Lemma 1.1.2. In particular,

$$|S_i| = |E| = j \qquad (i = 0, 1, \cdots, mn).$$

Thus $S_0, S_1, \cdots, S_{mn} \in \mathcal{A}_j$; and since $\sigma_A^n \circ \varphi_{K,A}(E) = E$, we have $S_{mn} = S_0$ and

$$S_{jn+i} = \kappa_K^{m-jl}(S_i) \qquad (0 \leq j < m, 0 \leq i < n).$$

If we set $f_i = P_{i+1} \circ (P_i|_E)^{-1}$ for $i = 0, 1, \cdots, mn - 1$, then $f_i \in F(S_i, S_{i+1})$ for all i; hence Lemma 1.1.3 implies that $S_0 S_1 \cdots S_{mn} \in \mathcal{I}_j(mn)$. By the same lemma, we also have $F(S_i, S_{i+1}) = \{f_i\}$ for all i, and this implies that $A_j(S_i, S_{i+1}) \in \{-1, 1\}$ for all i, and that

$$\begin{aligned}\prod_{i=0}^{n-1} A_j(S_i, S_{i+1}) K_j^l(S_n, S_0) &= \operatorname{sgn}(\kappa_K^l \circ f_{n-1} \circ \cdots \circ f_0) \\ &= \operatorname{sgn}(\kappa_K^l \circ P_n \circ (P_0|_E)^{-1}).\end{aligned}$$

Since $P_n(y) = P_0(\sigma_A^n y)$ and since $P_n(\varphi_{K,A}^l y) = \kappa_K^l(P_n y)$ for all $y \in \mathsf{X}_A$,

$$\operatorname{sgn}(\sigma_A^n \circ \varphi_{K,A}^l|_E) = \operatorname{sgn}(P_0 \circ \sigma_A^n \circ \varphi_{K,A}^l \circ (P_0|_E)^{-1}) = \operatorname{sgn}(\kappa_K^l \circ P_n \circ (P_0|_E)^{-1}),$$

and this proves the first assertion.

To prove the second assertion, suppose that $S_0, \cdots, S_n \in \mathcal{A}_j$, $\prod_{i=0}^{n-1} A_j(S_i, S_{i+1}) \neq 0$, and that

$$S_{jn+i} = \kappa_K^{m-jl}(S_i) \qquad (0 < j < m, 0 \leq i < n) \qquad \text{and} \qquad S_{mn} = S_0,$$

then $S_0 \cdots S_{mn} \in \mathcal{I}_j(mn)$ by (1.10). By Lemma 1.1.3, there are functions $f_0, f_1, \cdots, f_{mn-1}$ such that $F(S_i, S_{i+1}) = \{f_i\}$ for all i. Lemma 1.1.5 implies that there is an $E \in \mathcal{C}(mn)$ such that

$$S_i = P_i(E) \qquad (0 \leq i \leq mn).$$

Since $\mathcal{C}(n \circ l) \subset \mathcal{C}(mn)$, the uniqueness of E is obtained by Lemma 1.1.5.
□

As in (1.7), we put

$$(\mathcal{A}_j)_{(k)} = \{S \in \mathcal{A}_j : \kappa_K^k(S) = S\};$$

and we denote by $(A_j)_{(k)}$ and $(K_j)_{(k)}$ the restrictions of A_j and K_j to $(\mathcal{A}_j)_{(k)}$, respectively.

COROLLARY 1.2.8. *Suppose that X is a sofic shift and that φ is an automorphism of (X, σ_X) of order m for some positive integer m. Then for each positive divisor k of m, there are square matrices A_j and K_j $(j = 1, 2, \cdots r)$ over $\mathbb{Z}$ such that*

$$p_{(n \circ l)*k}(X, \sigma_X, \varphi) = \sum_{j=1}^{r} (-1)^{j+1} \mathrm{tr}\left((A_j)_{(k)}^n (K_j)_{(k)}^l\right) \quad (n > 0,\ 0 \leq l < k).$$

PROOF. Suppose that $x \in X$ is fixed by both $\sigma_X^n \circ \varphi^l$ and φ^k:

$$(\sigma_X^n \circ \varphi^l)(x) = \varphi^k(x) = x.$$

Since $\sigma_A^n \circ \varphi_{K,A}^l(\mathcal{L}_\infty^{-1}(x)) = \varphi_{K,A}^k(\mathcal{L}_\infty^{-1}(x)) = \mathcal{L}_\infty^{-1}(x)$, there is an $E \subset \mathcal{L}_\infty^{-1}(x)$ such that

$$\sigma_A^n \circ \varphi_{K,A}^l(E) = \varphi_{K,A}^k(E) = E.$$

Thus the result follows by the same argument as above. □

COROLLARY 1.2.9. *If φ is an automorphism of a sofic shift (X, σ_X) of order m for some positive integer m, then there are square matrices A_j and K_j $(j = 1, \cdots, r)$ over $\mathbb{Z}$ such that*

$$\zeta_{(X,\sigma_X,\varphi)}(t) = \prod_{\substack{k|m \\ 1 \leq k \leq m}} \exp\left(g_k(t)\right),$$

where

$$g_k(t) = \sum_{n=1}^{\infty} \sum_{l=0}^{k-1} \sum_{j=1}^{r} \frac{(-1)^{j+1} \mathrm{tr}\left((A_j)_{(k)}^n (K_j)_{(k)}^l\right)}{kn} t^{kn}.$$

1.3. Flip Systems

In [**10**], the Lind zeta function for shift-flip systems of finite type has been computed, and in [**13**], the result has been extended to the case of sofic shift-flip systems. In this section, we investigate the works in [**10**] and [**13**].

Let D_∞ be the infinite dihedral group generated by a and b such that

$$ab = ba^{-1} \qquad \text{and} \qquad b^2 = 1.$$

A flip system can be regarded as the D_∞-action $\alpha_2 : D_\infty \times X \to X$ on X given by

$$\alpha_2(a, x) = T(x) \qquad \text{and} \qquad \alpha_2(b, x) = F(x) \qquad (x \in X).$$

LEMMA 1.3.1. *All the finite index subgroups of D_∞ are as follows:*
(1) $\langle a^n \rangle \qquad (n > 0)$,
(2) $\langle a^n,\, a^j b \rangle \qquad (n > 0 \text{ and } 0 \leq j < n)$.

PROOF. Suppose that K is a finite index subgroup of D_∞. Then K must contain a^i for some integer $i \neq 0$. It follows that K is generated by a^n for some integer $n \neq 0$ or generated by a^n and $a^j b$ for some integers n and j with $n \neq 0$. Since $\langle a^n \rangle = \langle a^{-n} \rangle$ and $\langle a^n, a^j b \rangle = \langle a^n, a^{j-n} b \rangle$ we have the desired result by the division algorithm. □

REMARK. The indexes of these subgroups are as follows:
(1) $|D_\infty / \langle a^n \rangle| = 2n \qquad (n > 0)$,
(2) $|D_\infty / \langle a^n,\, a^j b \rangle| = n \qquad (n > 0 \text{ and } 0 \leq j < n)$.

If n is a positive integer and $j = 0, 1, \cdots, n-1$, then $p_{n,j}(X, T, F)$ will denote the number of points in X fixed by both T^n and $T^j \circ F$:

$$p_{n,j}(X, T, F) = |\{x \in X : T^n(x) = T^j \circ F(x) = x\}|.$$

It is clear that $p_{n,j}(X, T, F) = p_{n,n+j}(X, T, F)$. Since $TF = FT^{-1}$, $p_{n,j}(X, T, F) = p_{n,j+2}(X, T, F)$. Hence $p_{n,j}(X, T, F) = p_{n,0}(X, T, F)$ if n and j are even; $p_{n,j}(X, T, F) = p_{n,1}(X, T, F)$ if n is even and j is odd; and $p_{n,j}(X, T, F) =$

$p_{n,0}(X,T,F)$ for all j if n is odd. Thus, we have

$$\sum_{j=0}^{n-1} \frac{p_{n,j}(X,T,F)}{n} = \begin{cases} p_{n,0}(X,T,F) & \text{if } n \text{ is odd,} \\ \left(p_{n,0}(X,T,F) + p_{n,1}(X,T,F)\right)/2 & \text{if } n \text{ is even.} \end{cases} \tag{1.12}$$

The Lind zeta function $\zeta_{\alpha_2} = \zeta_{(X,T,F)}$ is given by

$$\zeta_{(X,T,F)}(t) = \sqrt{\zeta_{(X,T)}(t^2)}\, \exp\left(G_{(X,T,F)}(t)\right),$$

where $\zeta_{(X,T)}(t)$ is the Artin-Mazur zeta function of (X,T), and $G_{(X,T,F)}$ is given by

$$G_{(X,T,F)}(t) = \sum_{n=1}^{\infty} \Big(p_{2n-1,0}(X,T,F) t^{2n-1} \\ + \frac{p_{2n,0}(X,T,F) + p_{2n,1}(X,T,F)}{2} t^{2n} \Big).$$

The function $G_{(X,T,F)}(t)$ will be called the *generating function of a flip system* (X,T,F).

We will express $G_{(X,T,F)}(t)$ in terms of matrices when X is a sofic shift. We start with some preliminaries. A flip φ for a shift dynamical system (X,σ_X) is said to be a *one-block flip* if

$$x, x' \in X \ \text{ and } \ x_0 = x'_0 \quad \Rightarrow \quad \varphi(x)_0 = \varphi(x')_0.$$

In this case, there is a unique map $\tau : \mathcal{B}_1(X) \to \mathcal{B}_1(X)$ such that $\tau^2 = \mathrm{id}_{\mathcal{B}_1(X)}$ and that

$$\varphi(x)_i = \tau(x_{-i}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

We call τ the *symbol map* of φ.

Suppose that $\mathcal{A}$ is a finite set, and A, J are zero-one $\mathcal{A} \times \mathcal{A}$ matrices such that

$$AJ = JA^{\mathsf{T}} \qquad \text{and} \qquad J^2 = I. \tag{1.13}$$

Since J is zero-one and $J^2 = I$, it follows that there is a unique map $\tau_J : \mathcal{A} \to \mathcal{A}$ such that $\tau_J^2 = \mathrm{id}_{\mathcal{A}}$, and that for $a, b \in \mathcal{A}$ we have $J(a,b) = 1$ if and only if $\tau_J(a) = b$; and since $JA = A^{\mathsf{T}}J$, we have

$$A(a,b) = A(\tau_J(b), \tau_J(a)) \qquad (a, b \in \mathcal{A}). \tag{1.14}$$

We define $\widetilde{\varphi_J} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\widetilde{\varphi_J}(x)_i = \tau_J(x_{-i}) \qquad (x \in \mathcal{A}^{\mathbb{Z}};\ i \in \mathbb{Z}).$$

Since $\tau_J^2 = \mathrm{id}_{\mathcal{A}}$, it follows that $\widetilde{\varphi_J}$ is a one-block flip for $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ whose symbol map is τ_J; and (1.14) implies that $\widetilde{\varphi_J}(\mathsf{X}_A) = \mathsf{X}_A$. Thus the restriction of $\widetilde{\varphi_J}$ to X_A is a flip for (X_A, σ_A). We denote the restriction by $\widetilde{\varphi_{J,A}}$. A pair (A, J) of zero-one square matrices satisfying (1.13) will be called a *flip pair*.

The following is proved in [**10**].

LEMMA 1.3.2. *If (X, σ_X, φ) is a shift-flip system of finite type, then there is a flip pair (A, J) such that $(X, \sigma_X, \varphi) \cong (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$.*

Recall that the numbers of fixed points $p_{n,\delta}(X, \sigma_X, \varphi)$ $(n > 0, \delta \in \{0,\ 1\})$ are conjugacy invariant. So we will compute the generating function of $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$. To do this, we need some notation: if M is a matrix, $\mathcal{S}[M]$ will denote the sum of the entries of M, that is,

$$\mathcal{S}[M] = \sum_{I,J} M(I,J),$$

and if M is a square matrix, M^{Δ} will denote the diagonal matrix whose diagonal entries are identical with those of M, that is,

$$(M^{\Delta})(I,J) = \begin{cases} M(I,J) & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, $\mathrm{tr}(M) = \mathcal{S}[M^{\Delta}]$. The adjugate of a square matrix M will be denoted by $M^{\star}$ so that $MM^{\star} = (\det M)I$.

THEOREM 1.3.3. *If (A, J) is a flip pair, then*

$$\begin{aligned} p_{2n-1,0}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) &= \mathcal{S}[J^{\Delta}A^{n-1}(AJ)^{\Delta}], \\ p_{2n,0}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) &= \mathcal{S}[J^{\Delta}A^{n}J^{\Delta}] \quad \textit{and} \\ p_{2n,1}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) &= \mathcal{S}[(JA)^{\Delta}A^{n-1}(AJ)^{\Delta}] \qquad (n = 1, 2, \cdots). \end{aligned}$$

PROOF. For notational ease, we will write $\tau_J(a) = a^*$ for all $a \in \mathcal{A}$. It is easy to show that if $\sigma_A^{2n-1}(x) = \widetilde{\varphi_{J,A}}(x) = x$, then

$$x_{[0,2n-1]} = x_0 x_1 \cdots x_{n-1} x_{n-1}^* \cdots {x_1}^* \quad \text{and} \quad x_0^* = x_0;$$

if $\sigma_A^{2n}(x) = \widetilde{\varphi_{J,A}}(x) = x$, then

$$x_{[0,2n]} = x_0 x_1 \cdots x_{n-1} x_n x_{n-1}^* \cdots {x_1}^*, \; x_0^* = x_0, \text{ and } x_n^* = x_n;$$

and that if $\sigma_A^{2n}(x) = \sigma_A \circ \widetilde{\varphi_{J,A}}(x) = x$, then

$$x_{[0,2n]} = x_0 x_1 \cdots x_{n-1} x_{n-1}^* \cdots x_1^* x_0^*.$$

From (1.14), we have

$$\begin{aligned} &p_{2n-1,0}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \\ &\qquad = |\{x_0 \cdots x_{n-1} \in \mathcal{B}_n(\mathsf{X}_A) : x_0^* = x_0 \text{ and } A(x_{n-1}, x_{n-1}^*) = 1\}|, \\ &p_{2n,0}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \\ &\qquad = |\{x_0 \cdots x_n \in \mathcal{B}_{n+1}(\mathsf{X}_A) : x_0^* = x_0 \text{ and } x_n^* = x_n\}|, \end{aligned}$$

and

$$\begin{aligned} &p_{2n,0}(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \\ &\qquad = |\{x_0 \cdots x_{n-1} \in \mathcal{B}_n(\mathsf{X}_A) : A(x_0^*, x_0) = A(x_{n-1}, x_{n-1}^*) = 1\}|. \end{aligned}$$

Since for each $a \in \mathcal{A}$, a^* is the unique element of $\mathcal{A}$ such that

$$J(a, a^*) = J(a^*, a) = 1,$$

it follows that

$$A(a, b^*) = AJ(a,b) \quad \text{and} \quad A(a^*, b) = JA(a,b) \qquad (a, b \in \mathcal{A}),$$

and the result follows. □

Let s be a complex number such that its modulus $|s|$ is less than $1/\Lambda(A)$, where $\Lambda(A)$ is the spectral radius of A. Then we have

$$\sum_{i=0}^{\infty} s^i A^i = (I - sA)^{-1} = \frac{1}{\det(I - sA)} (I - sA)^{\star}.$$

From this, we obtain the following.

COROLLARY 1.3.4. *If (A, J) is a flip pair, then*

$$G_{(\mathsf{X}_A,\sigma_A,\widetilde{\varphi_{J,A}})}(t) = \frac{1}{\det(I - t^2A)}\mathcal{S}\Big[tJ^\Delta(I - t^2A)^\star(AJ)^\Delta$$
$$+\frac{t^2}{2}\{J^\Delta A(I - t^2A)^\star J^\Delta + (JA)^\Delta(I - t^2A)^\star(AJ)^\Delta\}\Big]$$

for $t^2 < 1/\Lambda(A)$.

Theorem 1.3.3 is extended to the sofic case in [**13**]:

THEOREM 1.3.5. *If (X, σ_X, φ) is a sofic shift-flip system, then there are square matrices A_j, B_j and J_j, $j = 1, 2, \cdots, r$, over $\mathbb{Z}$ such that*

$$\begin{aligned}
p_{2n-1,0}(X,\sigma_X,\varphi) &= \sum_{j=1}^{r}(-1)^{j+1}\,\mathcal{S}\big[J_j^\Delta B_j^{n-1}(A_jJ_j)^\Delta\big],\\
p_{2n,0}(X,\sigma_X,\varphi) &= \sum_{j=1}^{r}(-1)^{j+1}\,\mathcal{S}\big[J_j^\Delta B_j^{n}J_j^\Delta\big] \quad \textit{and}\\
p_{2n,1}(X,\sigma_X,\varphi) &= \sum_{j=1}^{r}(-1)^{j+1}\,\mathcal{S}\big[(J_jA_j)^\Delta B_j^{n-1}(A_jJ_j)^\Delta\big]\ (n = 1, 2, \cdots).
\end{aligned}$$

The following is proved in [**13**] (see also Lemma 1.4.3).

LEMMA 1.3.6. *Let (X, σ_X, φ) be a shift-flip system. Then there is a shift-flip system (Y, σ_Y, ψ) such that it is conjugate to (X, σ_X, φ) and ψ is one-block.*

Let X be a sofic shift and φ be a one-block flip for (X, σ_X) with symbol map τ. Suppose that a finite set $\mathcal{A}$, a map (labeling) $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ and zero-one $\mathcal{A} \times \mathcal{A}$ matrices A and J having the following properties:
(1) $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring.
(2) (A, J) is a flip pair.
(3) $\mathcal{L} \circ \tau_J = \tau \circ \mathcal{L}$.
In this case, the quadruple $(\mathcal{A}, \mathcal{L}, A, J)$ will be called a *presentation for sofic shift-flip system* (X, σ_X, φ). Note that (3) implies $\mathcal{L}_\infty \circ \widetilde{\varphi_{J,A}} = \varphi \circ \mathcal{L}_\infty$.

The following proposition is proved in [**13**] (see also §3.1).

PROPOSITION 1.3.7. *Suppose that X is a sofic shift and that φ is a one-block flip for (X, σ_X) with symbol map τ. Then there is a presentation $(\mathcal{A}, \mathcal{L}, A, J)$ for (X, σ_X, φ) having the following properties:*
(4) $\mathcal{L}_\infty$ has no graph diamonds.
(5) If $\delta \in \{0, 1\}$, $x \in X$ and $\sigma_X^\delta \circ \varphi(x) = x$, then there is a $y \in \mathsf{X}_A$ such that $\mathcal{L}_\infty(y) = x$ and $\sigma_A^\delta \circ \widetilde{\varphi_{J,A}}(y) = y$.

To prove Theorem 1.3.5, suppose that (X, σ_X, φ) is a sofic shift-flip system. By Lemma 1.3.6, we may assume that φ is a one-block flip. Let τ denote the symbol map of φ, and let $(\mathcal{A}, \mathcal{L}, A, J)$ be a presentation satisfying (4) and (5) of Proposition 1.3.7.

As in §1.1, we denote the cardinality of $\mathcal{A}$ by r, the projection $\mathsf{X}_A \ni y \mapsto y_i \in \mathcal{A}$ by P_i for each integer i, and we fix a linear order $<$ of $\mathcal{A}$. For each positive integer $j \leq r$, we set

$$\mathcal{A}_j = \{S \subset \mathcal{A} : |S| = j \quad \text{and} \quad |\mathcal{L}(S)| = 1\};$$

for $S_1, S_2 \in \mathcal{A}_j$, we denote by $F(S_1, S_2)$ the set of one-to-one functions $f : S_1 \to S_2$ such that $A(a, f(a)) = 1$ for all $a \in S_1$.

If $S \subset \mathcal{A}$, the set $\tau_J(S)$ will be denoted by S^*. It is obvious that $(S^*)^* = S$. When $S, T \subset \mathcal{A}$ and $f : S \to T$ is a one-to-one correspondence, we denote the map

$$T^* \ni \tau_J(f(a)) \mapsto \tau_J(a) \qquad (a \in S)$$

by f^*. Thus $f^* = (\tau_J|_S) \circ f^{-1} \circ (\tau_J|_{T^*}) : T^* \to S^*$ is a one-to-one correspondence, and we have

$$\operatorname{sgn}(f^*) = \operatorname{sgn}(\tau_J|_S)\operatorname{sgn}(f^{-1})\operatorname{sgn}(\tau_J|_{T^*}). \tag{1.15}$$

It is clear that $(f^*)^* = f$.

If j is a positive integer with $j \leq r$, then we set

$$\mathcal{A}_j = \{S \subset \mathcal{A} : |S| = j \text{ and } |\mathcal{L}(S)| = 1\};$$

and for $S_1, S_2 \in \mathcal{A}_j$ we denote by $F(S_1, S_2)$ the set of one-to-one functions $f : S_1 \to S_2$ such that $A(a, f(a)) = 1$ for all $a \in S_1$. By (4) in Proposition

1.3.7, we have $S^* \in \mathcal{A}_j$ whenever $S \in \mathcal{A}_j$; and, by (1.14), we have $f^* \in F(S_2^*, S_1^*)$ whenever $f \in F(S_1, S_2)$. It is clear that

$$F(S_1, S_2) \ni f \mapsto f^* \in F(S_2^*, S_1^*)$$

is a one-to-one correspondence.

For notational convenience, we adopt the convention that an empty sum is 0 and an empty product is 1:

$$\sum_{i \in \varnothing} x_i = 0 \qquad \text{and} \qquad \prod_{i \in \varnothing} x_i = 1.$$

Define the $\mathcal{A}_j \times \mathcal{A}_j$ matrices A_j, B_j and J_j by

$$A_j(S_1, S_2) = \sum_{f \in F(S_1, S_2)} \operatorname{sgn}(f),$$

$$B_j(S_1, S_2) = \begin{cases} 1 & \text{if } F(S_1, S_2) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$J_j(S_1, S_2) = \begin{cases} \operatorname{sgn}(\tau_J|_{S_1}) & \text{if } \tau_J(S_1) = S_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that J_j is symmetric and satisfies $J_j^2 = I$. A simple calculation using (1.15) and the fact that $F(S_1, S_2) \ni f \mapsto f^* \in F(S_2^*, S_1^*)$ is a one-to-one correspondence gives

$$A_j(S_2^*, S_1^*) = J_j(S_1^*, S_1) A_j(S_1, S_2) J_j(S_2, S_2^*) \quad (S_1, S_2 \in \mathcal{A}_j), \tag{1.16}$$

which may be expressed as $A_j^\top = J_j A_j J_j$, or equivalently, as $J_j A_j = A_j^\top J_j$. Similarly, we have $B_j(S_1, S_2) = B_j(S_2^*, S_1^*)$.

Let m be a positive integer. As in §1.1, we denote by $\mathcal{I}_j(m)$ the set of all $(m+1)$-blocks $S_0 S_1 \cdots S_m$ over the alphabet $\mathcal{A}_j$ such that $S_0 = S_m$ and $\Pi_{i=0}^{m-1} A_j(S_i, S_{i+1}) \neq 0$.

The following is the modification of Lemma 1.1.4.

LEMMA 1.3.8. *Let π be a permutation of a finite set S. Suppose $\mathcal{C} \subset 2^S$ and $S_0 \subset S$ satisfy the following:*
(1) *$\pi(E) = E$ for every $E \in \mathcal{C}$.*
(2) *If $E_1, E_2 \in \mathcal{C}$, then $E_1 \cup E_2, E_1 \setminus E_2 \in \mathcal{C}$.*
(3) *$S_0 \neq \varnothing$ and $\pi(S_0) = S_0$.*
(4) *If $E \subset S_0$ and $\pi(E) = E$, then $E \in \mathcal{C}$.*
Then

$$\sum_{E \in \mathcal{C} \setminus \{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\pi|_E) = 1. \tag{1.17}$$

PROOF. The assumptions imply that $\mathcal{C}$ has at least two distinct elements, namely S_0 and $\varnothing$. For convenience, we set $\operatorname{sgn}(\pi|_\varnothing) = 1$. With this notation, (1.17) is equivalent to

$$\sum_{E \in \mathcal{C}} (-1)^{|E|} \operatorname{sgn}(\pi|_E) = 0.$$

If we put

$$\mathcal{N} = \{E \in \mathcal{C} : E \subset S_0\} \text{ and } \mathcal{R} = \{E \in \mathcal{C} : E \cap S_0 = \varnothing\},$$

then (2) implies that

$$\mathcal{N} \times \mathcal{R} \ni (E_1, E_2) \mapsto E_1 \cup E_2 \in \mathcal{C}$$

is a one-to-one correspondence. If $(E_1, E_2) \in \mathcal{N} \times \mathcal{R}$, then $E_1 \cap E_2 = \varnothing$, and hence

$$(-1)^{|E_1 \cup E_2|} \operatorname{sgn}(\pi|_{E_1 \cup E_2}) = (-1)^{|E_1|} \operatorname{sgn}(\pi|_{E_1}) \, (-1)^{|E_2|} \operatorname{sgn}(\pi|_{E_2}).$$

From (3) and Lemma 1.1.4 we have

$$\sum_{E \subset S_0, \, \pi(E) = E} (-1)^{|E|} \operatorname{sgn}(\pi|_E) = 0;$$

and from (1) and (4), we have

$$\{E \subset S_0 : \pi(E) = E\} = \mathcal{N}.$$

Therefore

$$\begin{aligned}\sum_{E\in\mathcal{C}}(-1)^{|E|}\operatorname{sgn}(\pi|_E) &= \sum_{(E_1,E_2)\in\mathcal{N}\times\mathcal{R}}(-1)^{|E_1\cup E_2|}\operatorname{sgn}(\pi|_{E_1\cup E_2})\\ &= \sum_{E_1\in\mathcal{N}}(-1)^{|E_1|}\operatorname{sgn}(\pi|_{E_1})\sum_{E_2\in\mathcal{R}}(-1)^{|E_2|}\operatorname{sgn}(\pi|_{E_2})\\ &= 0.\end{aligned}$$

□

REMARK. The condition that $S_0\neq\varnothing$ is crucial. Otherwise we cannot apply Lemma 1.1.4.

Now, suppose that N is a positive integer and $\delta\in\{0,1\}$. We put

$$F(N,\delta)=\{x\in X:\sigma_X^N(x)=\sigma_X^\delta\,\varphi(x)=x\},$$

so that $p_{N,\delta}(\sigma_X,\varphi)=|F(N,\delta)|$; and for $x\in F(N,\delta)$ we put

$$\mathcal{C}(N,\delta;\,x)=\{E\subset\mathcal{L}_\infty^{-1}(x):\sigma_A^N(E)=\sigma_A^\delta\,\varphi_{J,A}(E)=E\}.$$

It is clear that $F(N,\delta)\subset F(N)$ and that $\mathcal{C}(N,\delta;\,x)\subset\mathcal{C}(N;\,x)$ for each $x\in F(N,\delta)$, where

$$F(N)=\{x\in X:\sigma_X^N(x)=x\}$$

and

$$\mathcal{C}(N;\,x)=\{E\subset\mathcal{L}_\infty^{-1}(x):\sigma_A^N(E)=E\}.$$

LEMMA 1.3.9. *We have*

$$p_{N,\delta}(\sigma_X,\varphi)=\sum_{x\in F(N,\delta)}\ \sum_{E\in\mathcal{C}(N,\delta;\,x)\setminus\{\varnothing\}}(-1)^{|E|+1}\operatorname{sgn}(\sigma_A^N|_E).$$

PROOF. Let $x\in F(N,\delta)$ and put

$$S_0(x)=\{y\in\mathcal{L}_\infty^{-1}(x):\exists l\in\mathbb{Z}\ \ \sigma_A^\delta\,\varphi_{J,A}(y)=\sigma_A^{lN}(y)\}.$$

It follows from (5) in Proposition 1.3.7 that $S_0(x)\neq\varnothing$. It is then straightforward to verify that the conditions in Lemma 1.3.8 are satisfied with

$S = \mathcal{L}_\infty^{-1}(x)$, $\pi = \sigma_A^N|_{\mathcal{L}_\infty^{-1}(x)}$, $\mathcal{C} = \mathcal{C}(N,\delta;\,x)$ and $S_0 = S_0(x)$. Hence we obtain

$$\sum_{E\in\mathcal{C}(N,\delta;\,x)\setminus\{\varnothing\}} (-1)^{|E|+1}\operatorname{sgn}(\sigma_A^N|_E) = 1 \qquad (x \in F(N,\delta)),$$

and the result follows. □

If we set

$$\mathcal{C}(N,\delta) = \bigcup_{x\in F(N,\delta)} \mathcal{C}(N,\delta;\,x)$$

and

$$\mathcal{C}_j(N,\delta) = \{E \in \mathcal{C}(N,\delta) : |E| = j\} \qquad (j = 1, 2, \cdots, r),$$

then, as in the proof of Theorem 1.1.1, we have

$$p_{N,\delta}(\sigma_X,\varphi) = \sum_{j=1}^{r} (-1)^{j+1} \sum_{E\in\mathcal{C}_j(N,\delta)} \operatorname{sgn}(\sigma_A^N|_E).$$

Therefore Theorem 1.3.5 will follow, once we prove that

$$\sum_{E\in\mathcal{C}_j(2n-1,0)} \operatorname{sgn}(\sigma_A^{2n-1}|_E) = \mathcal{S}\big[J_j^\Delta B_j^{n-1}(A_jJ_j)^\Delta\big], \tag{1.18}$$

$$\sum_{E\in\mathcal{C}_j(2n,0)} \operatorname{sgn}(\sigma_A^{2n}|_E) = \mathcal{S}\big[J_j^\Delta B_j^{n} J_j^\Delta\big], \tag{1.19}$$

and

$$\sum_{E\in\mathcal{C}_j(2n,1)} \operatorname{sgn}(\sigma_A^{2n}|_E) = \mathcal{S}\big[(J_jA_j)^\Delta B_j^{n-1}(A_jJ_j)^\Delta\big] \tag{1.20}$$

for $n = 1, 2, \cdots$ and for $j = 1, 2, \cdots, r$.

We prove (1.18) only. The others are similarly proved. Let n and j be given. First of all, it is easy to prove that

$$\begin{aligned} &\mathcal{C}_j(2n-1,0) \\ &= \{E \in \mathcal{C}_j(2n-1) : P_{n+i}(E) = P_{n-i-1}(E)^* \quad (0 \le i < n)\}. \end{aligned} \tag{1.21}$$

In particular, every $E \in \mathcal{C}_j(2n-1,0)$ is completely determined by $P_0(E)$, $\cdots$, $P_{n-1}(E)$. A simple calculation gives

$$\begin{aligned}&\mathcal{S}\big[J_j^{\Delta} B_j^{n-1}(A_j J_j)^{\Delta}\big]\\&= \sum_{S_0\cdots S_{n-1}\in\mathcal{A}_j^n} J_j(S_0,S_0)\Big[\prod_{i=0}^{n-2} B_k(S_i,S_{i+1})\Big] A_j(S_{n-1},S_{n-1}^*)J_j(S_{n-1}^*,S_{n-1}).\end{aligned}$$

If $S_0\cdots S_{n-1}\in\mathcal{A}_j^n$, then the corresponding term in the right hand side is different from zero if and only if

$$S_0 = S_0^*,\ B_j(S_i,S_{i+1}) = 1\ \ (0 \le i \le n-2),\ \text{and}\ A_j(S_{n-1},S_{n-1}^*) \ne 0. \tag{1.22}$$

In this case, the term reduces to

$$J_j(S_0,S_0)A_j(S_{n-1},S_{n-1}^*)J_j(S_{n-1}^*,S_{n-1}).$$

If we denote the set of all n-blocks $S_0\cdots S_{n-1}$ over $\mathcal{A}_j$ that satisfy (1.22) by $\mathcal{J}_j(2n-1,0)$, then we have

$$\begin{aligned}&\mathcal{S}\big[J_j^{\Delta} B_j^{n-1}(A_j J_j)^{\Delta}\big]\\&= \sum_{S_0\cdots S_{n-1}\in\mathcal{J}_j(2n-1,0)} J_j(S_0,S_0)A_j(S_{n-1},S_{n-1}^*)J_j(S_{n-1}^*,S_{n-1}).\end{aligned}$$

Thus (1.18) is a consequence of the following:

Lemma 1.3.10. *If $E\in\mathcal{C}_j(2n-1,0)$ and*

$$S_i = P_i(E) \qquad (0 \le i \le n-1), \tag{1.23}$$

then, $S_0\cdots S_{n-1}\in\mathcal{J}_j(2n-1,0)$ and

$$\operatorname{sgn}(\sigma_A^{2n-1}|_E) = J_j(S_0,S_0)A_j(S_{n-1},S_{n-1}^*)J_j(S_{n-1}^*,S_{n-1}). \tag{1.24}$$

Conversely, if $S_0\cdots S_{n-1}\in\mathcal{J}_j(2n-1,0)$, then there is a unique $E\in\mathcal{C}_j(2n-1,0)$ such that (1.23) *holds.*

Proof. Suppose $E\in\mathcal{C}_j(2n-1,0)$ and S_0, S_1, $\cdots$, S_{n-1} are given by (1.23). Then $E\in\mathcal{C}_j(2n-1)$ and Lemma 1.1.5 implies that

$$P_0(E)P_1(E)\cdots P_{2n-1}(E)\in\mathcal{I}_j(2n-1),$$

$A_j(P_i(E), P_{i+1}(E)) \in \{1, -1\}$ for $0 \le i \le 2n-2$, and

$$\operatorname{sgn}(\sigma_A^{2n-1}|_E) = \prod_{i=0}^{2n-2} A_j(P_i(E), P_{i+1}(E)).$$

By (1.21) and (1.23), we have

$$P_0(E)P_1(E)\cdots P_{2n-1}(E) = S_0 \cdots S_{n-1}S_{n-1}^* \cdots S_0^*.$$

Hence $S_0 \cdots S_{n-1} \in \mathcal{J}_j(2n-1, 0)$, and we have

$$\operatorname{sgn}(\sigma_A^{2n-1}|_E) = \Big[\prod_{i=0}^{n-2} A_j(S_i, S_{i+1})\Big] A_j(S_{n-1}, S_{n-1}^*)\Big[\prod_{i=0}^{n-2} A_j(S_{i+1}^*, S_i^*)\Big]. \tag{1.25}$$

If $n = 1$, the right hand side of (1.25) is equal to $A_j(S_0, S_0^*)$ and that of (1.24) is equal to $J_j(S_0, S_0)A_j(S_0, S_0^*)J_j(S_0^*, S_0)$, hence (1.24) holds, because $S_0 = S_0^*$ and $J_j(S, S^*) = J_j(S^*, S) \in \{-1, 1\}$ for all $S \in \mathcal{A}_j$. Now suppose that $n \ge 2$. Then (1.16) implies that

$$\prod_{i=0}^{n-2} A_j(S_{i+1}^*, S_i^*) = \prod_{i=0}^{n-2} \big(J_j(S_i^*, S_i)A_j(S_i, S_{i+1})J_j(S_{i+1}, S_{i+1}^*)\big)$$

and the right hand side reduces to

$$J_j(S_0^*, S_0)\Big[\prod_{i=0}^{n-2} A_j(S_i, S_{i+1})\Big] J_j(S_{n-1}, S_{n-1}^*),$$

because $J_j(S, S^*) = J_j(S^*, S) \in \{-1, 1\}$ for all $S \in \mathcal{A}_j$. Hence the right hand side of (1.25) is equal to that of (1.24), because $A_j(S_i, S_{i+1}) \in \{-1, 1\}$ for all i. This proves (1.24).

Conversely, suppose that $S_0 \cdots S_{n-1} \in \mathcal{J}_j(2n-1, 0)$. Then

$$S_0 \cdots S_{n-1}S_{n-1}^* \cdots S_0^* \in \mathcal{I}_j(2n-1),$$

by Lemma 1.1.3; and Lemma 1.1.5 implies that there is an $E \in \mathcal{C}_j(2n-1)$ such that

$$P_0(E)P_1(E)\cdots P_{2n-1}(E) = S_0 \cdots S_{n-1}S_{n-1}^* \cdots S_0^*.$$

Thus $E \in \mathcal{C}_j(2n-1,0)$, by (1.21). Finally, the uniqueness is obvious, because every $E \in \mathcal{C}_j(2n-1,0)$ is completely determined by $P_0(E)\cdots P_{n-1}(E)$. $\square$

COROLLARY 1.3.11. *If (X, σ_X, φ) is a sofic shift-flip system, then there are square matrices A_j, B_j and J_j, $j = 1, 2, \cdots, r$, over $\mathbb{Z}$ such that*

$$G_{(X,\sigma_X,\varphi)}(t) = \sum_{j=1}^{r} (-1)^{j+1} \frac{1}{\det(I - t^2 B_j)} \mathcal{S}\Big[t J_j^{\Delta}(I - t^2 B_j)^{\star}(A_j J_j)^{\Delta}$$
$$+ \frac{t^2}{2}\{J_j^{\Delta} B_j (I - t^2 B_j)^{\star} J_j^{\Delta} + (J_j A_j)^{\Delta}(I - t^2 B_j)^{\star}(A_j J_j)^{\Delta}\}\Big]$$

for $t^2 < 1/\Lambda(B_1)$.

1.4. Reversals of Finite Order

If m is a positive integer, then G_{2m} will denote the group generated by two elements a and b such that

$$ab = ba^{-1} \qquad \text{and} \qquad b^{2m} = 1.$$

If (X, T, R) is a reversal system of order $2m$, then we define $\alpha_{2m} : G_{2m} \times X \to X$ by

$$\alpha_{2m}(a, x) = T(x) \qquad \text{and} \qquad \alpha_{2m}(b, x) = R(x).$$

If $m = 1$, then G_2 is the infinite dihedral group D_∞.

LEMMA 1.4.1. *If m is a positive integer greater than 1, then all the finite index subgroups of G_{2m} are as follows:*

(1) $\langle a^n \rangle \qquad (n > 0)$,
(2) $\langle a^n b^{2l} \rangle \qquad (n > 0$ and $0 < l < m)$,
(3) $\langle a^n,\, b^{2k} \rangle \qquad (n > 0,\ 0 < k < m,$ and $k|m)$,
(4) $\langle a^n b^{2l},\, b^{2k} \rangle \qquad (n > 0,\ 0 < l < k < m,$ and $k|m)$,
(5) $\langle a^n,\, a^j b^{2k-1} \rangle \qquad (n > 0,\ 0 \le j < n$ and $2k-1|m)$.

PROOF. By Lemma 1.2.2, (1)-(4) are obtained, since those are all the finite index subgroup of $\langle a, b^2 \rangle$. So we consider the finite index subgroup K

of G_{2m} which contains $a^i b^{2l-1}$ for some integers i and l with $0 < 2l-1 < 2m$. Suppose that l and k are positive integers such that

$$0 < 2l-1,\ 2k-1 < 2m \quad \text{and} \quad \gcd(2l-1, 2m) = 2k-1.$$

Then $a^i b^{2l-1}$ generates $\langle a^i b^{2k-1}\rangle$. Since $\langle a^n, a^j b^{2k-1}\rangle = \langle a^n, a^{j-n} b^{2k-1}\rangle$, we obtain (5) by the division algorithm.

Suppose that $K = \langle a^n b^{2l},\ a^j b^{2k-1}\rangle$ for some integers n, l, j, and k with $n > 0$, $0 < l < m$, $0 < 2k-1 < m$, and $2k-1|m$. Let d be the greatest common divisor of $2l$ and $2k-1$. It is obvious that d is odd and that $K \subset \langle a^n, a^j b^d\rangle$. It is easy to show that $K \supset \langle a^n, a^j b^d\rangle$ and this is the case (5). $\square$

Remark. The indexes of these subgroups are as follows:

(1) $|G_{2m}/\langle a^n\rangle| = 2mn \qquad (n > 0)$,
(2) $|G_{2m}/\langle a^n c^l\rangle| = 2mn \qquad (n > 0 \text{ and } 0 < l < m)$,
(3) $|G_{2m}/\langle a^n,\ c^k\rangle| = 2kn \qquad (n > 0,\ 0 < k < m, \text{ and } k|m)$,
(4) $|G_{2m}/\langle a^n c^l,\ c^k\rangle| = 2kn \qquad (n > 0,\ 0 < l < k < m, \text{ and } k|m)$,
(5) $|G_{2m}/\langle a^n,\ a^j b^{2k-1}\rangle| = (2k-1)n \quad (n > 0,\ 0 \le j < n \text{ and } 2k-1|m)$.

Suppose that (X, T, R) is a reversal system of order $2m$ and that x is fixed by all elements of $\langle a^n,\ a^j b^{2k-1}\rangle$:

$$T^n(x) = T^j \circ R^{2k-1}(x) = x.$$

Then we have

$$R^{4k-2}(x) = R^{2k-1} \circ T^{-j} \circ T^j \circ R^{2k-1}(x) = (T^j \circ R^{2k-1})^2(x) = x.$$

If we put $X_{(4k-2)} = \{x \in X : R^{4k-2}(x) = x\}$, then R^{2k-1} is a flip map for $(X_{(4k-2)}, T)$.

The Lind zeta function for a reversal system (X, T, R) of order $2m$ is given by

$$\zeta_{(X,T,R)}(t) = \sqrt{\zeta_{(X,T,R^2)}(t^2)} \prod_{\substack{2k-1|m \\ 1 \le 2k-1 \le m}} \exp\left(\frac{G_{4k-2}(t^{2k-1})}{2k-1}\right), \tag{1.26}$$

where $\zeta_{(X,T,R^2)}(t)$ is the Lind zeta function for automorphism R^2 of order m and $G_{4k-2}(t)$ is the generating function of a flip system $(X_{(4k-2)}, T, R^{2k-1})$.

We have already known how to compute $\zeta_{(X,T,R^2)}(t)$ and $G_{4k-2}(t)$ when (X, T, R) is a sofic shift-reversal system. But it would be a long calculation if we compute them separately. To save labor, we discuss the existence of matrices which have the information about automorphisms R^2 and the flip map R^{2k-1}. We first treat shift-reversal systems of finite type.

Let $\mathcal{A}$ be a finite set and (X, σ_X) a shift space over $\mathcal{A}$. A reversal φ for (X, σ_X) is said to be a *one-block reversal* if

$$x, x' \in X \quad \text{and} \quad x_0 = x'_0 \quad \Rightarrow \quad \varphi(x)_0 = \varphi(x')_0.$$

The order of a one-block reversal must be finite, because φ is a homeomorphism and $\mathcal{A}$ is finite. Suppose that φ is a one-block reversal of order $2m$. Then there is a unique map $\tau : \mathcal{A} \to \mathcal{A}$ such that

$$\tau^{2m} = \mathrm{id}_{\mathcal{A}} \quad \text{and} \quad \varphi(x)_i = \tau(x_{-i}) \qquad (x \in X;\, i \in \mathbb{Z}).$$

We call τ the symbol map of φ.

Suppose that $\mathcal{A}$ is a finite set again, m is a positive integer, and A, J are zero-one $\mathcal{A} \times \mathcal{A}$ matrices such that

$$AJ = JA^{\mathsf{T}}, \qquad J^{2m} = I, \qquad \text{and} \qquad J^{2k} \neq I \quad (0 < k < m). \tag{1.27}$$

Since J is zero-one and $J^{2m} = I$, it follows that there is a unique map $\tau_J : \mathcal{A} \to \mathcal{A}$ such that $\tau_J^{2m} = \mathrm{id}_{\mathcal{A}}$, and that for $a, b \in \mathcal{A}$ we have $J(a, b) = 1$ if and only if $\tau_J(a) = b$; and since $AJ = JA^{\mathsf{T}}$, we have

$$A(a, b) = A(\tau_J(b), \tau_J(a)) \qquad (a, b \in \mathcal{A}), \tag{1.28}$$

and hence

$$A(\tau_J^{2k}(a), \tau_J^{2k}(b)) = A(\tau_J^{2k+1}(b), \tau_J^{2k+1}(a)) \quad (a, b \in \mathcal{A};\ k = 0, \cdots, m-1).$$

We define $\widetilde{\varphi_J} : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\widetilde{\varphi_J}(x)_i = \tau_J(x_{-i}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

Since $\tau_J^{2m} = \mathrm{id}_{\mathcal{A}}$, $\widetilde{\varphi_J}$ is a one-block reversal for $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ whose symbol map is τ_J; and (1.28) implies that $\widetilde{\varphi_J}(\mathsf{X}_A) = \mathsf{X}_A$. Thus the restriction of $\widetilde{\varphi_J}$ to X_A is a reversal for (X_A, σ_A). We denote the restriction by $\widetilde{\varphi_{J,A}}$. The pair (A, J) of $\mathcal{A} \times \mathcal{A}$ zero-one matrices satisfying (1.27) for some positive integer m will be called a *reversal pair*.

The following is proved in [**14**].

LEMMA 1.4.2. *If X is a shift of finite type and φ is a reversal for (X, σ_X), then there is a reversal pair (A, J) such that $(X, \sigma_X, \varphi) \cong (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$.*

Suppose that (A, J) is a reversal pair satisfying (1.27). We write $K = J^2$ and $\kappa_K = \tau_J^2$ so that

$$AK = KA, \qquad K^m = I, \qquad \text{and} \qquad K^i \neq I \qquad (0 < i < m).$$

Then the Lind zeta function for $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ is given by

$$\zeta_{(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})}(t) = \sqrt{\zeta_{(\mathsf{X}_A, \sigma_A, \varphi_{K,A})}(t^2)} \prod_{\substack{2k-1 | m \\ 1 \leq 2k-1 \leq m}} \exp\left(\frac{G_{4k-2}(t^{2k-1})}{2k-1}\right),$$

where $G_{4k-2}(t)$ is the generating function of a flip system determined by a flip pair $\left(A_{(4k-2)}, J_{(4k-2)}^{2k-1}\right)$.

The following lemma is the generalization of Lemma 1.3.6.

LEMMA 1.4.3. *Let (X, σ_X, φ) be a shift-reversal system. If the order of φ is finite, then there is a shift-reversal system (Y, σ_Y, ψ) such that it is conjugate to (X, σ_X, φ) and ψ is a one-block reversal.*

PROOF. We assume that $|\varphi| = 2m$. We define $\mathcal{A}$, $\tau : \mathcal{A} \to \mathcal{A}$, $\Psi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$, and $\theta : X \to \mathcal{A}^{\mathbb{Z}}$ by

$$\mathcal{A} = \{(a_0, a_1, \cdots, a_{2m-1}) : \exists x \in X \ a_k = \varphi^k(x)_0 \text{ for } k = 0, \cdots, 2m-1\},$$

$$\tau(a_0, a_1, \cdots, a_{2m-1}) = (a_1, \cdots, a_{2m-1}, a_0) \qquad ((a_0, a_1, \cdots, a_{2m-1}) \in \mathcal{A}),$$

$$\Psi(y)_i = \tau(y_{-i}) \qquad (y \in \mathcal{A}^{\mathbb{Z}};\ i \in \mathbb{Z}),$$

and

$$\theta(x)_i = (b_0, b_1, \cdots, b_{2m-1}),$$

where for all $k = 0, \cdots, 2m-1$, $b_k = \varphi^k(x)_i$ if k is even, and $b_k = \varphi^k(x)_{-i}$ if k is odd:

$$\theta(x)_i = (x_i, \varphi(x)_{-i}, \varphi^2(x)_i, \cdots, \varphi^{2m-2}(x)_i, \varphi^{2m-1}(x)_{-i}) \quad (x \in X;\ i \in \mathbb{Z}).$$

Then $\mathcal{A}$ is a finite set, $\tau^{2m} = \mathrm{id}_{\mathcal{A}}$, Ψ is a one-block reversal for $(\mathcal{A}^{\mathbb{Z}}, \sigma)$, θ is one-to-one and continuous, and we have

$$\theta \circ \sigma_X = \sigma \circ \theta \qquad \text{and} \qquad \theta \circ \varphi = \Psi \circ \theta;$$

hence we have the desired result by setting $Y = \theta(X)$ and $\psi = \Psi|_Y$. □

Proposition 1.3.7 plays a crucial role when we compute generating functions of sofic shift-flip systems. We generalize it to the reversal cases as follows:

PROPOSITION 1.4.4. *Let X be a sofic shift and φ be a one-block reversal for (X, σ_X) with symbol map τ. There is a presentation $(\mathcal{A}, \mathcal{L}, A, J)$ for (X, σ_X, φ) having the following properties:*
(4) $\mathcal{L}_\infty$ has no graph diamonds.
(5) If $\delta \in \{0, 1\}$, $x \in X$ and $\sigma_X^\delta \circ \varphi^i(x) = x$ for a positive odd divisor i of m, then there is a $y \in \mathsf{X}_A$ such that $\mathcal{L}_\infty(y) = x$ and $\sigma_A^\delta \circ \widetilde{\varphi_{J,A}}^i(y) = y$.

We will prove the above generalization in §3.1.

Suppose that $(\mathcal{A}, \mathcal{L}, A, J)$ is a presentation for sofic shift-reversal system (X, σ_X, φ) of order $2m$ and that $(\mathcal{A}, \mathcal{L}, A, J)$ satisfies (4) and (5) of Proposition 4.4. When m is odd and $2k-1$ is a positive divisor of m, $(\mathcal{A}_{(4k-2)}, \mathcal{L}, A_{(4k-2)}, J_{(4k-2)}^{2k-1})$ is a presentation for the sofic-shift flip system $(X_{(4k-2)}, \sigma_X, \varphi^{2k-1})$ satisfying (4) and (5) of Proposition 1.3.7.

1.5. ℕ-Rationality

Let $\mathbb{N}[t]$ be the set of all polynomials over $\mathbb{N}$ in the variable t. It is clear that $\mathbb{N}[t]$ is closed under the operations sum and product:

$$f(t), g(t) \in \mathbb{N}[t] \qquad \Rightarrow \qquad (f+g)(t), (fg)(t) \in \mathbb{N}[t].$$

We define the formal operation star as follows:

$$f^*(t) = \frac{1}{1-f(t)} = \sum_{n=0}^{\infty} f^n(t) \qquad (f(t) \in \mathbb{N}[t]).$$

(Here, $f^n(t)$ is the function obtained by product of f n-times.) The closure of $\mathbb{N}[t]$ is the set which is closed under finite numbers of sum, product, and star. A series $f(t)$ is called $\mathbb{N}$-rational if it is an element of the closure of $\mathbb{N}[t]$. For instance, let $\mathcal{A}$ be a finite set and let X be the full $\mathcal{A}$-shift; if we denote the cardinality of $\mathcal{A}$ by r, then the function

$$\sum_{n=1}^{\infty} |\mathcal{B}_n(X)| t^n = \sum_{n=1}^{\infty} r^n t^n = \frac{1}{1-rt}$$

is $\mathbb{N}$-rational.

If $L \subset \bigcup_{n=1}^{\infty} \mathcal{A}^n$ is a language, then we define the star operation as follows:

$$L^* = \bigcup_{n=1}^{\infty} L^n.$$

A language $L \subset \bigcup_{n=1}^{\infty} \mathcal{A}^n$ is called *rational* if it is obtained from finite languages by a finite number of union, product, and star. A language L is called *cyclic* if

$$uv \in L \quad \Leftrightarrow \quad vu \in L$$

and

$$w \in L \quad \Leftrightarrow \quad w^n \in L \qquad (n \in \mathbb{N}).$$

It is well known [**2**] that if L is rational, then

$$\sum_{n=1}^{\infty} |L \cap \mathcal{A}^n| \, t^n$$

is $\mathbb{N}$-rational. It is also well-known [**2**] that if L is rational and cyclic, then

$$\exp\left(\sum_{n=1}^{\infty} \frac{|L \cap \mathcal{A}^n|}{n} \, t^n\right)$$

is ℕ-rational. For instance, if X is a sofic shift, then the set

$$\bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\{w^i \in \mathcal{B}_{in}(X) : \exists x \in X\ x_{[0,n-1]} = w \text{ and } \sigma_X^n(x) = x\}$$

is rational and cyclic [**6**]. From this, the Artin-Mazur zeta function of a sofic shift (X, σ_X) is ℕ-rational.

1.5.1. Automorphisms of sofic shifts of finite order. Let $\mathcal{A}$ be a finite set, X a sofic shift over $\mathcal{A}$, and φ an automorphism of (X, σ_X) of prime order m. The set L_l given by

$$L_l = \bigcup_{i=1}^{\infty}\bigcup_{n=1}^{\infty}\{w^i \in \mathcal{B}_{imn}(X) : \exists x \in X\ x_{[0,mn-1]} = w \text{ and } \sigma_X^n \circ \varphi^l(x) = x\}$$

is rational and cyclic. Thus

$$\exp\left(\sum_{n=1}^{\infty}\frac{p_{nol}(X,\sigma_X,\varphi)}{mn}t^{mn}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{|L_l \cap \mathcal{A}^n|}{mn}t^{mn}\right)$$

is ℕ-rational for all $l = 0, 1, \cdots, m-1$. We observe that the Lind zeta function of (X, σ_X, φ) is given by products of $(m+1)$ ℕ-rational functions:

$$\zeta_{(X,\sigma_X,\varphi)}(t) = \prod_{l=0}^{m-1}\exp\left(\sum_{n=1}^{\infty}\frac{p_{nol}(X,\sigma_X,\varphi)}{mn}t^{mn}\right)\zeta_{(X_{(1)},T)}(t),$$

where $\zeta_{(X_{(1)},T)}$ is the Artin-Mazur zeta function of $(X_{(1)}, T)$. Thus $\zeta_{(X,\sigma_X,\varphi)}$ is ℕ-rational.

When m is an arbitrary positive integer, the similar argument gives ℕ-rationality of the Lind zeta function for automorphisms of sofic shifts of order m.

1.5.2. Generating functions of sofic shifts-flip system.

Proposition 1.5.1. *Suppose that (X, T, F) is a flip system. If n is a positive integer and $p_{2n,0}(X,T,F) + p_{2n,1}(X,T,F) < \infty$, then $p_{2n,0}(X,T,F) + p_{2n,1}(X,T,F)$ is even.*

PROOF. We put

$$A = \{x \in X : T^{2n}x = Fx = x,\ T \circ Fx \neq x\},$$

$$B = \{x \in X : T^{2n}x = T \circ Fx = x,\ Fx \neq x\},$$

and

$$C = \{x \in X : T^{2n}x = Fx = T \circ Fx = x\}.$$

It is obvious that A, B, C are mutually disjoint, and that

$$p_{2n,0}(X,T,F) + p_{2n,1}(X,T,F) = |A| + |B| + 2|C|. \tag{1.29}$$

For $x \in A \cup B$ we denote the least period of x by $q(x)$ and define Ix by

$$Ix = \begin{cases} T^{q(x)/2}x & \text{if } q(x) \text{ is even,} \\ T^{(q(x)+1)/2}x & \text{if } q(x) \text{ is odd and } x \in A, \\ T^{(q(x)-1)/2}x & \text{if } q(x) \text{ is odd and } x \in B. \end{cases}$$

Then $I : A \cup B \to A \cup B$ is well defined; and satisfies

$$I^2x = x \quad \text{and} \quad Ix \neq x \qquad (x \in A \cup B).$$

Hence $|A \cup B|$ is even, and the result follows from (1.29). □

Now, suppose that (X, σ_X, φ) is a sofic shift-flip system. Then the sets

$$\bigcup_{n=1}^{\infty} \{x_{[0,n-1]} : x \in X \text{ and } \sigma_X^{2n-1}x = \varphi x = x\},$$

$$\bigcup_{n=1}^{\infty} \{x_{[0,n]} : x \in X \text{ and } \sigma_X^{2n}x = \varphi x = x\}$$

and

$$\bigcup_{n=1}^{\infty} \{x_{[0,n-1]} : x \in X \text{ and } \sigma_X^{2n}x = \sigma_X \circ \varphi x = x\}$$

are rational ([**6**], pp. 14, 55). Hence their generating functions

$$\sum_{n=1}^{\infty} p_{2n-1,0}(\sigma_X, \varphi)t^n, \quad \sum_{n=1}^{\infty} p_{2n,0}(\sigma_X, \varphi)t^{n+1} \quad \text{and} \quad \sum_{n=1}^{\infty} p_{2n,1}(\sigma_X, \varphi)t^n$$

are $\mathbb{N}$-rational and consequently, so is $2G_{\sigma_X,\varphi}$. Finally, Berstel's theorem [**2, 6, 21**] and Soittola's theorem [**2, 21**] imply that $G_{(X,\sigma_X,\varphi)}$ is $\mathbb{N}$-rational.

1.5.3. Sofic shift-Reversal systems of finite order. Suppose that (X, σ_X, φ) is a sofic shift-reversal system of order $2m$ for some positive integer m. The following example shows that neither $\sqrt{\zeta_{(X,\sigma_X,\varphi^2)}(t^2)}$ nor $\left(G_{4k-2}(t^{2k-1})\right)/(2k-1)$ is $\mathbb{N}$-rational in general.

EXAMPLE 1.5.1. Let A and J be zero-one matrices given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then (A, J) is a reversal pair and $\widetilde{\varphi_{J,A}}$ is a reversal for (X_A, σ_A) of order 6.

We first compute the Lind zeta function for the automorphism $\kappa_\infty|_{\mathsf{X}_A}$, where $\kappa = \tau_J^2$. It is easy to show that $A^n = A^n J^2 = A^n J^4$ for all positive integers n, and it is obvious that $A_{(1)} = [1]$. Thus we have

$$g_3(t) = \sum_{n=1}^{\infty} \sum_{l=0}^{2} \frac{\operatorname{tr}\left(A^n J^{2l}\right)}{3n} t^{3n} = \sum_{n=1}^{\infty} \frac{\operatorname{tr}(A^n)}{n} t^{3n},$$

and

$$g_1(t) = \sum_{n=1}^{\infty} \frac{\operatorname{tr}\left(A_{(1)}^n\right)}{n} t^n = \sum_{n=1}^{\infty} \frac{t^n}{n};$$

and the Lind zeta function for $(\mathsf{X}_A, \sigma_A, \kappa_\infty|_{\mathsf{X}_A})$ is given by

$$\zeta_{(\mathsf{X}_A,\sigma_A,\kappa_\infty|_{\mathsf{X}_A})}(t) = \exp(g_3(t)) \exp(g_1(t)) = \frac{1}{(1-t)(1-t^3-6t^6-6t^9)}.$$

On the other hand, if we put $L = J^3$ and $A_{(1)} = J_{(1)} = [1]$, then the generating functions $G_3(t)$ for the flip system $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{L,A}})$ and $G_1(t)$ for $(\mathsf{X}_{A_{(1)}}, \sigma_{A_{(1)}}, \widetilde{\varphi_{J_{(1)},A_{(1)}}})$ are given by

$$G_3(t) = \frac{t + t^2 + 3t^4 + 3t^6}{1 - t^2 - 6t^4 - 6t^6} \qquad \text{and} \qquad G_1(t) = \frac{t}{1-t}.$$

Now, (1.26) gives that the Lind zeta function for $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ as follows:

$$\sqrt{\frac{1}{(1-t^2)(1-t^6-6t^{12}-6t^{18})}}\exp\left(\frac{t^3+t^6+3t^{12}+3t^{18}}{3-3t^6-18t^{12}-18t^{18}}+\frac{t}{1-t}\right).$$

We observe that

$$\sqrt{\frac{1}{(1-t^2)(1-t^6-6t^{12}-6t^{18})}} \quad \text{and} \quad \frac{t^3+t^6+3t^{12}+3t^{18}}{3-3t^6-18t^{12}-18t^{18}}+\frac{t}{1-t}$$

are not $\mathbb{N}$-rational [**21**].

CHAPTER 2

Williams' Decomposition Theorem for Sofic Shift-Reversal Systems of Finite Order

Let us recall the concepts of elementary equivalence, strong shift equivalence and shift equivalence. Suppose that A and B are zero-one square matrices. A pair (D, E) of zero-one matrices satisfying

$$A = DE \qquad \text{and} \qquad B = DE$$

is said to be an *elementary equivalence from A to B*. If there is an elementary equivalence (D, E) form A to B, then we write $(D, E) : A \approx B$. If $(D, E) : A \approx B$, we define the block map $\Gamma_{D,E} : \mathcal{B}_2(\mathsf{X}_A) \to \mathcal{B}_1(\mathsf{X}_B)$ by

$$\Gamma_{D,E}(a_1 a_2) = b,$$

where $a_1 a_2 \in \mathcal{B}_2(\mathsf{X}_A)$ and $b \in \mathcal{B}_1(\mathsf{X}_B)$ satisfy $D(a_1, b) = E(b, a_2) = 1$. Since $A = DE$, $B = ED$, and since A, B, D and E are zero-one matrices, $\Gamma_{D,E}$ is well defined. The map $\gamma_{D,E} : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_B, \sigma_B)$ defined by

$$\gamma_{D,E}(x)_i = \Gamma(x_{[i,i+1]}) \qquad (x \in \mathsf{X}_A;\ i \in \mathbb{Z}),$$

is called an *elementary conjugacy.*

A *strong shift equivalence of lag n from A to B* is a sequence of elementary equivalences

$$\begin{aligned}
(D_1, E_1) &: A \approx A_2 \\
(D_2, E_2) &: A_2 \approx A_3 \\
&\ \ \vdots \\
(D_n, E_n) &: A_n \approx B.
\end{aligned}$$

If there is a strong shift equivalence of lag n from A to B, then we write $A \approx B$ (lag n). It is evident that if $A \approx B$ (lag n) for some positive integer n, then (X_A, σ_A) is conjugate to (X_B, σ_B). Conversely, by Williams' decomposition theorem ([**22**], Chapter 7 of [**16**]) every conjugacy between shifts of finite type can be decomposed into the composition of a finite number of elementary conjugacies.

A pair (D, E) of zero-one matrices satisfying

$$A^n = DE, \quad B^n = ED, \quad AD = DB, \quad \text{and} \quad EA = BE,$$

for some positive integer n, is said to be a *shift equivalence of lag n from A to B*. If there is a shift equivalence of lag n from A to B then we write $(D, E) : A \sim B$ (lag n). It is evident that if $A \approx B$ (lag n), then (D, E) : $A \sim B$ (lag n). It is known [**11, 12**] that the converse is not true in general.

In [**19**], Nasu has introduced the notion of symbolic adjacency matrix (bipartite code) and he has extended the decomposition result to the sofic cases.

The purpose of this chapter is to establish decomposition results for shift-reversal systems of finite type and sofic shift-reversal systems. In §2.1, we first introduce the notions of half elementary equivalence and half strong shift-reversal equivalence and then prove that every conjugacy between shift-reversal systems of finite type can be decomposed into the composition of an even number of half elementary conjugacies. We extend in §2.2 the decomposition theorem to the case of sofic shift-reversal systems. In §2.3, we define a shift-reversal equivalence in a similar manner to a shift equivalence, and show that the existence of shift-reversal equivalence does not guarantee the existence of half strong shift-reversal equivalence.

2.1. Shift-Reversal Systems of Finite Type

Suppose that (A, J) and (B, K) are reversal pairs. A pair (D, E) of zero-one matrices satisfying

$$A = DE, \quad B = ED, \quad DK = JE^{\mathsf{T}}, \quad \text{and} \quad EJ = KD^{\mathsf{T}}$$

is said to be a *half elementary equivalence from* (A, J) *to* (B, K). If there is a half elementary equivalence from (A, J) to (B, K), then we write $(D, E) : (A, J) \approxeq (B, K)$. We note that $DK = JE^{\mathsf{T}}$ is equivalent to $EJ = KD^{\mathsf{T}}$ when (A, J) and (B, K) are flip pairs.

PROPOSITION 2.1.1. *Suppose that* (A, J) *and* (B, K) *are reversal pairs. If there is a half elementary equivalence* (D, E) *from* (A, J) *to* (B, K)*, then there is a conjugacy from* $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ *to* $(\mathsf{X}_B, \sigma_B, \sigma_B \circ \widetilde{\varphi_{K,B}})$*.*

PROOF. It is enough to show that the elementary equivalence $\gamma_{D,E} : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_B, \sigma_B)$ satisfies

$$\gamma_{D,E} \circ \widetilde{\varphi_{J,A}} = (\sigma_B \circ \widetilde{\varphi_{K,B}}) \circ \gamma_{D,E}.$$

Suppose that A and J are $\mathcal{A} \times \mathcal{A}$ matrices and that B and K are $\mathcal{A}' \times \mathcal{A}'$ matrices for some finite sets $\mathcal{A}$ and $\mathcal{A}'$. Since $DK = JE^{\mathsf{T}}$, it follows that

$$D(a, b) = 1 \quad \Leftrightarrow \quad E(\tau_K(b), \tau_J(a)) = 1 \qquad (a \in \mathcal{A}, b \in \mathcal{A}');$$

and since $EJ = KD^{\mathsf{T}}$, it follows that

$$E(b, a) = 1 \quad \Leftrightarrow \quad D(\tau_J(a), \tau_K(b)) = 1 \qquad (a \in \mathcal{A}, b \in \mathcal{A}').$$

Hence

$$\Gamma_{D,E}(a_1 a_2) = b \quad \Leftrightarrow \quad \Gamma_{D,E}(\tau_J(a_2)\tau_J(a_1)) = \tau_K(b) \quad (a_1 a_2 \in \mathcal{B}_2(\mathsf{X}_A)).$$

From this, we have $\gamma_{D,E} \circ \widetilde{\varphi_{J,A}} = (\sigma_B \circ \widetilde{\varphi_{K,B}}) \circ \gamma_{D,E}$. $\square$

The conjugacy $\gamma_{D,E} : (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \to (\mathsf{X}_B, \sigma_B, \sigma_B \circ \widetilde{\varphi_{K,B}})$ given by as above, will be called a *half elementary conjugacy*.

A sequence of n half elementary equivalences

$$\begin{aligned}
&(D_1, E_1) : (A, J) \approxeq (A_2, J_2), \\
&(D_2, E_2) : (A_2, J_2) \approxeq (A_3, J_3), \\
&\qquad \vdots \\
&(D_n, E_n) : (A_n, J_n) \approxeq (B, K).
\end{aligned}$$

is said to be a *half strong shift-reversal equivalence of lag n from (A, J) to (B, K)*. If there is a half strong shift-reversal equivalence of lag n from (A, J) to (B, K), then we write $(A, J) \approx (B, K)$ (lag n).

We consider the higher block system of a shift-reversal system. We need some notation. Suppose that $\mathcal{A}$ is a finite set, X is a shift space over $\mathcal{A}$, and that φ is a reversal for (X, σ_X) of order $2m$. We may assume φ is a one-block reversal by Lemma 1.4.3 in the previous chapter. Let $\tau : \mathcal{A} \to \mathcal{A}$ be the symbol map of φ, that is,

$$\varphi(x)_i = \tau(x_{-i}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

We define $\tau_k : \mathcal{A}^k \to \mathcal{A}^k$ and $\widetilde{\tau_k} : \mathcal{A}^k \to \mathcal{A}^k$ by

$$\tau_k(a_1 a_2 \cdots a_k) = \tau(a_1)\tau(a_2)\cdots\tau(a_k) \qquad (a_1 a_2 \cdots a_n \in \mathcal{A}^k),$$

and

$$\widetilde{\tau_k}(a_1 a_2 \cdots a_k) = \tau(a_k)\cdots\tau(a_2)\tau(a_1) \qquad (a_1 a_2 \cdots a_n \in \mathcal{A}^k).$$

It is obvious that $\widetilde{(\widetilde{\tau_k})} = \tau_k$. Let (X_k, σ_k) denote the k-th higher block system of X for a positive integer k. We define $\varphi_k : X_k \to X_k$ by

$$\varphi_k(x)_i = \widetilde{\tau_k}(x_{-i}) \qquad (x \in X_k;\ i \in \mathbb{Z}).$$

Then φ_k becomes a one-block reversal for (X_k, σ_k) of order $2m$ and $\widetilde{\tau_k}$ is the symbol map of φ_k. The shift-reversal system $(X_k, \sigma_k, \varphi_k)$ will be called a *k-th higher block shift-reversal system of* (X, σ_X, φ). When X is a shift of finite type, there is a reversal pair (A, J) and (A_k, J_k) such that $(X, \sigma_X, \varphi) \cong (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(X_k, \sigma_k, \varphi_k) \cong (\mathsf{X}_{A_k}, \sigma_{A_k}, \widetilde{\varphi_{J_k,A_k}})$ for all k by Lemma 1.4.2. The following lemma shows that there is a half strong shift-reversal equivalence from (A, J) to (A_k, J_k).

Lemma 2.1.2. *For each positive integer k, we have*

$$(A, J) \approx (A_{k+1}, J_{k+1}) \ (\mathrm{lag}\, k).$$

PROOF. We define zero-one matrices D_k and E_k by

$$D_k(u,v) = \begin{cases} 1 & \text{if } u = v_{[1,k]} \\ 0 & \text{otherwise} \end{cases} \qquad (u \in \mathcal{B}_k(\mathsf{X}_A), v \in \mathcal{B}_{k+1}(\mathsf{X}_A)),$$

and

$$E_k(v,u) = \begin{cases} 1 & \text{if } u = v_{[2,k+1]} \\ 0 & \text{otherwise} \end{cases} \qquad (u \in \mathcal{B}_k(\mathsf{X}_A), v \in \mathcal{B}_{k+1}(\mathsf{X}_A)).$$

It is easy to show that $(D_k, E_k) : (A_k, J_k) \approxeq (A_{k+1}, J_{k+1})$. $\square$

By Proposition 2.1.1, we have

$$(A,J) \approx (B,K) \text{ (lag } n) \quad \Rightarrow \quad (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \cong (\mathsf{X}_B, \sigma_B, {\sigma_B}^n \circ \widetilde{\varphi_{K,B}}).$$

Recall that if (X, T, R) is a reversal system, then T^n is a conjugacy from (X, T, R) to $(X, T, T^{2n} \circ R)$ for all positive integers n. Therefore, we have

$$(A,J) \approx (B,K) \text{ (lag } 2n) \quad \Rightarrow \quad (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \cong (\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$$

and

$$(A,J) \approx (B,K) \text{ (lag } (2n-1)) \quad \Rightarrow \quad (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \cong (\mathsf{X}_B, \sigma_B, \sigma_B \circ \widetilde{\varphi_{K,B}})$$

for all positive integers n.

The following theorem is the main result of this section.

THEOREM 2.1.3. *If* $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \cong (\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$, *then* $(A,J) \approx (B,K)$ (lag n) *for some even number* n.

In order to prove Theorem 2.1.3, we need some notation. Let X be a shift space and n be a positive integer. When $w = a_1 a_2 \cdots a_n \in \mathcal{B}_n(X)$, we denote the initial and the terminal symbols of w by $i(w)$ and $t(w)$, respectively:

$$i(w) = a_1 \qquad \text{and} \qquad t(w) = a_n;$$

and if $n = 1$, then $i(w) = t(w) = w$.

Suppose that (A, J) and (B, K) are reversal pairs. Let (A_k, J_k) and (B_k, K_k) denote reversal pairs such that $(\mathsf{X}_{A_k}, \sigma_{A_k}, \widetilde{\varphi_{J_k,A_k}})$ and $(\mathsf{X}_{B_k}, \sigma_{B_k},$

$\widetilde{\varphi_{K_k,B_k}}$) are conjugate to the k-th higher block shift-reversal systems of $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$, respectively.

Now we prove Theorem 2.1.3.

PROOF. Suppose that $\theta : (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \to (\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$ is a conjugacy with memory s and anticipation t, that is, there is a block map $\Theta : \mathcal{B}_{s+t+1}(\mathsf{X}_A) \to \mathcal{B}_1(\mathsf{X}_B)$ such that

$$\theta(x)_i = \Theta(x_{[i-s,i+t]}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

By Lemma 2.1.2,

$$(A, J) \approx (A_{s+t+1}, J_{s+t+1})\ (\mathrm{lag}(s+t)).$$

We may assume that $s + t$ is even and it is enough to prove the theorem in the case of $s = t = 0$.

Suppose that $\kappa : \mathcal{B}_1(\mathsf{X}_A) \to \mathcal{B}_1(\mathsf{X}_B)$ is a block map such that $\theta = \kappa_\infty$ and that θ^{-1} is the inverse of θ with memory r and anticipation n. If necessary, by extending window size, we may assume that $r = n$.

Let $\mathcal{A}_i$ $(i = 1, 2, \cdots, 2r + 1)$ be the finite set given by

$$\mathcal{A}_i = \{(u, w, v) \in \mathcal{B}_j(\mathsf{X}_B) \times \mathcal{B}_l(\mathsf{X}_A) \times \mathcal{B}_j(\mathsf{X}_B) : u\kappa_l(w)v \in \mathcal{B}_1(\mathsf{X}_{B_i})\},$$

where $j = \lfloor \frac{i-1}{2} \rfloor$ and $l = i - 2\lfloor \frac{i-1}{2} \rfloor$ ($n \in \mathbb{Z}$ and $n \le x < n + 1 \Rightarrow \lfloor x \rfloor = n$.) We define $\mathcal{A}_i \times \mathcal{A}_i$ matrices C_i and L_i as follows: if (u_1, w_1, v_1), (u_2, w_2, v_2) $\in \mathcal{A}_i$, then

$$C_i\left((u_1, w_1, v_1), (u_2, w_2, v_2)\right) = \begin{cases} 1 & \text{if } (u_1\kappa_l(w_1)v_1)(u_2\kappa_l(w_2)v_2) \in \mathcal{B}_2(\mathsf{X}_{B_i}) \text{ and } w_1w_2 \in \mathcal{B}_2(\mathsf{X}_{A_l}) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$L_i\left((u_1, w_1, v_1), (u_2, w_2, v_2)\right) = \begin{cases} 1 & \text{if } u_2 = \widetilde{\tau_{K_j}}(v_1),\ w_2 = \widetilde{\tau_{J_l}}(w_1),\ \text{and } v_2 = \widetilde{\tau_{K_j}}(u_1) \\ 0 & \text{otherwise.} \end{cases}$$

Then (C_i, L_i) is a reversal pair. If we define $\mathcal{A}_i \times \mathcal{A}_{i+1}$ matrix F_i and $\mathcal{A}_{i+1} \times \mathcal{A}_i$ matrix G_i by

$$F_i((u_1, w_1, v_1), (u_2, w_2, v_2)) = \begin{cases} 1 & \text{if } u_1\kappa(w_1)v_1 = (u_2\kappa(w_2)v_2)_{[1,i]}, \\ & \quad \text{and } t(w_1) = i(w_2) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$G_i((u_2, w_2, v_2), (u_1, w_1, v_1)) = \begin{cases} 1 & \text{if } (u_2\kappa(w_2)v_2)_{[2,i+1]} = u_1\kappa(w_1)v_1 \\ & \quad \text{and } t(w_1) = i(w_2) \\ 0 & \text{otherwise,} \end{cases}$$

for $(u_1, w_1, v_1) \in \mathcal{A}_i$ and $(u_2, w_2, v_2) \in \mathcal{A}_{i+1}$, then we obtain

$$(F_i, G_i) : (C_i, L_i) \approxeq (C_{i+1}, L_{i+1}).$$

For convenience, we write $C = C_{2r+1}$ and $L = L_{2r+1}$. Since $C_1 = A$ and $L_1 = J$, we have

$$(A, J) \approx (C, L) \text{ (lag } 2r).$$

On the other hand, by recoding of symbols, $(\mathsf{X}_C, \sigma_C, \widetilde{\varphi_{L,C}})$ is equal to the $(2r+1)$-th higher block shift-reversal system of $(\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$. Therefore, we have

$$(B, K) \approx (C, L) \text{ (lag } 2r),$$

by Lemma 2.1.2, and hence

$$(A, J) \approx (B, K) \text{ (lag } 4r).$$

$\square$

2.2. Sofic-Shift Reversal Systems

Suppose that (X, σ_X, φ) and (Y, σ_Y, ψ) are sofic shift-reversal systems. We may assume that φ and ψ are one-block reversal by Lemma 1.4.3. Let τ be a symbol map of φ:

$$\varphi(x)_i = \tau(x_{-i}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

Recall that a presentation $(\mathcal{A}, \mathcal{L}, A, J)$ for (X, σ_X, φ) is a quadruple of a finite set $\mathcal{A}$, a labeling $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$, and zero-one $\mathcal{A} \times \mathcal{A}$ matrices A and J such that
(1) $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring,
(2) (A, J) is a reversal pair, and
(3) $\mathcal{L} \circ \tau_J = \tau \circ \mathcal{L}$.
Throughout the section, we assume that $(\mathcal{C}, \mathcal{L}, A, J)$ and $(\mathcal{D}, \mathcal{K}, B, K)$ are presentations for (X, σ_X, φ) and (Y, σ_Y, ψ), respectively.

We introduce some notation. If k and l are positive integers, we define $\mathcal{L}_k : \mathcal{B}_k(\mathsf{X}_A) \to \mathcal{B}_k(X)$ and $\mathcal{L}_k \times \mathcal{K}_l : \mathcal{B}_k(\mathsf{X}_A) \times \mathcal{B}_l(\mathsf{X}_B) \to \mathcal{B}_k(X) \times \mathcal{B}_l(Y)$ by

$$\mathcal{L}_k(a_1 \cdots a_k) = \mathcal{L}(a_1) \cdots \mathcal{L}(a_k) \qquad (a_1 \cdots a_k \in \mathcal{B}_l(\mathsf{X}_A))$$

and

$$(\mathcal{L}_k \times \mathcal{K}_l)(u, v) = (\mathcal{L}_k(u), \mathcal{K}_l(v)) \qquad (u \in \mathcal{B}_k(\mathsf{X}_A),\ v \in \mathcal{B}_l(\mathsf{X}_B)).$$

A half elementary equivalence (D, E) from (A, J) to (B, K) will be called a *labeled half elementary equivalence* (D, E) *from* $(\mathcal{C}, \mathcal{L}, A, J)$ *to* $(\mathcal{D}, \mathcal{K}, B, K)$, provided it has the following properties:
(1) if $a_1a_2, a_3a_4 \in \mathcal{B}_2(\mathsf{X}_A)$ and $\mathcal{L}_2(a_1a_2) = \mathcal{L}_2(a_3a_4)$, then

$$\mathcal{K} \circ \Gamma_{D,E}(a_1a_2) = \mathcal{K} \circ \Gamma_{D,E}(a_3a_4),$$

and
(2) if $b_1b_2, b_3b_4 \in \mathcal{B}_2(\mathsf{X}_B)$ and $\mathcal{K}_2(b_1b_2) = \mathcal{K}_2(b_3b_4)$, then

$$\mathcal{L} \circ \Gamma_{E,D}(b_1b_2) = \mathcal{L} \circ \Gamma_{E,D}(b_3b_4).$$

If (D, E) is a labeled half elementary equivalence from $(\mathcal{C}, \mathcal{L}, A, J)$ to $(\mathcal{D}, \mathcal{K}, B, K)$, then we write $(D, E) : (\mathcal{C}, \mathcal{L}, A, J) \approxeq (\mathcal{D}, \mathcal{K}, B, K)$.

Proposition 2.2.1. *If If (D, E) is a labeled half elementary equivalence from $(\mathcal{C}, \mathcal{L}, A, J)$ to $(\mathcal{D}, \mathcal{K}, B, K)$, then there is a conjugacy $\lambda_{D,E}$ from (X, σ_X, φ) to $(Y, \sigma_Y, \sigma_Y \circ \psi)$ such that*

$$\lambda_{D,E} \circ \mathcal{L}_\infty = \mathcal{K}_\infty \circ \gamma_{D,E}. \tag{2.1}$$

PROOF. Let $\Lambda_{D,E} : \mathcal{B}_2(X) \to \mathcal{B}_1(Y)$ be a block map defined by

$$\Lambda_{D,E}(x_1x_2) = \mathcal{K} \circ \Gamma_{D,E}(a_1a_2) \qquad (x_1x_2 \in \mathcal{B}_2(X); a_1a_2 \in \mathcal{L}_2^{-1}(x_1x_2)).$$

It is obvious that $\Lambda_{D,E}$ is well-defined. We define $\lambda_{D,E} : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \sigma_Y \circ \psi)$ by

$$\lambda_{D,E}(x)_i = \Lambda_{D,E}(x_{[i,i+1]}) \qquad (x \in X;\ i \in \mathbb{Z}).$$

It is easy to show that $\lambda_{D,E}$ is a conjugacy satisfying (2.1). □

The conjugacy $\lambda_{D,E} : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \sigma_Y \circ \psi)$ defined by as above, will be called a *labeled half elementary conjugacy.*

A sequence of n labeled half elementary equivalences

$$\begin{aligned}
&(D_1, E_1) : (\mathcal{C}, \mathcal{L}, A, J) \approxeq (\mathcal{C}^{(2)}, \mathcal{L}^{(2)}, A^{(2)}, J^{(2)}),\\
&(D_2, E_2) : (\mathcal{C}^{(2)}, \mathcal{L}^{(2)}, A^{(2)}, J^{(2)}) \approxeq (\mathcal{C}^{(3)}, \mathcal{L}^{(3)}, A^{(3)}, J^{(3)}),\\
&\qquad \vdots\\
&(D_n, E_n) : (\mathcal{C}^{(n)}, \mathcal{L}^{(n)}, A^{(n)}, J^{(n)}) \approxeq (\mathcal{D}, \mathcal{K}, B, K).
\end{aligned}$$

is said to be a *half strong shift-reversal equivalence of lag n from $(\mathcal{C}, \mathcal{L}, A, J)$ to* $(\mathcal{D}, \mathcal{K}, B, K)$. If there is a half strong shift-reversal equivalence of lag n from $(\mathcal{C}, \mathcal{L}, A, J)$ to $(\mathcal{D}, \mathcal{K}, B, K)$, then we write $(\mathcal{C}, \mathcal{L}, A, J) \approx (\mathcal{D}, \mathcal{K}, B, K)$ (lag n).

Suppose that there is a half strong shift-reversal equivalence of lag n from $(\mathcal{C}, \mathcal{L}, A, J)$ to $(\mathcal{D}, \mathcal{K}, B, K)$ for some even number n. It is obvious that there is a conjugacy $\eta : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \psi)$ and $\theta : (\mathsf{X}_A, \sigma_A, \varphi_{J,A}) \to (\mathsf{X}_B, \sigma_B, \varphi_{K,B})$ such that

$$\eta \circ \mathcal{L}_\infty = \mathcal{K}_\infty \circ \theta.$$

We consider the higher block system of a sofic shift-reversal system (X, σ_X, φ), where φ is a one-block reversal. Suppose that $(\mathcal{C}, \mathcal{L}, A, J)$ is a presentation for (X, σ_X, φ). As in §2.1, let (A_k, J_k) denote a reversal pair such that $(\mathsf{X}_{A_k}, \sigma_{A_k}, \widetilde{\varphi_{J_k,A_k}})$ is conjugate to the k-th higher block shift-reversal system of $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ for all k. Then $(\mathcal{B}_k(X), \mathcal{L}_k, A_k, J_k)$ is a presentation for some sofic shift-reversal system which is conjugate to the k-th higher block shift-reversal system of (X, σ_X, φ).

LEMMA 2.2.2. *For each positive integer k, we have*

$$(\mathcal{C}, \mathcal{L}, A, J) \approx (\mathcal{B}_k(X), \mathcal{L}_k, A_k, J_k) \ (\operatorname{lag} k).$$

PROOF. We define D_k and E_k as in the proof of Lemma 2.1.2. It is obvious that (D_k, E_k) is a labeled half elementary equivalence from $(\mathcal{B}_k(\mathsf{X}_A)$, $\mathcal{L}_k$, A_k, $J_k)$ to $(\mathcal{B}_{k+1}(\mathsf{X}_A), \mathcal{L}_{k+1}, A_{k+1}, J_{k+1})$ for all positive integers k. $\square$

The following is the main result of this section.

THEOREM 2.2.3. *Suppose that $\eta : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \psi)$ is a conjugacy. Then η can be decomposed into the composition of an even number of labeled half elementary conjugacies.*

In order to prove Theorem 2.2.3, we need the following proposition.

PROPOSITION 2.2.4. *Suppose that $\eta : (X, \sigma_X, \varphi) \to (Y, \sigma_Y, \psi)$ is a conjugacy. Then there are presentations $(\mathcal{C}, \mathcal{L}, A, J)$ for (X, σ_X, φ) and $(\mathcal{D}, \mathcal{K}, B, K)$ for (Y, σ_Y, ψ), and a conjugacy*

$$\theta : (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \to (\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$$

such that

$$\eta \circ \mathcal{L}_\infty = \mathcal{K}_\infty \circ \theta. \tag{2.2}$$

We assume Proposition 2.2.4 and prove Theorem 2.2.3. Proposition 2.2.4 will be proved in the next chapter.

PROOF. Suppose that $\theta : (\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}}) \to (\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$ is a conjugacy satisfying (2.2). As in the proof of Theorem 2.1.3, we may assume θ is a one-block conjugacy and θ^{-1} is the inverse of θ with memory r and anticipation r.

We define F_i and G_i as in the proof of Theorem 2.1.3 for $i = 1, 2, \cdots, 2r+1$. It is easy to show that

$$(F_i, G_i) : (\mathcal{A}_i, \mathcal{J}_i, C_i, L_i) \approxeq (\mathcal{A}_{i+1}, \mathcal{J}_{i+1}, C_{i+1}, L_{i+1}),$$

where $\mathcal{J}_i = \mathcal{K}_j \times \mathcal{L}_l \times \mathcal{K}_j$, $j = \lfloor \frac{i-1}{2} \rfloor$, and $l = i - 2j$ for all i. By Lemma 2.2.2 and the same argument as in the proof of Theorem 2.1.3, the result follows. $\square$

2.3. Shift-Reversal Equivalence

Suppose that A and B are zero-one square matrices. Recall that $(\mathsf{X}_A, \sigma_A) \cong (\mathsf{X}_B, \sigma_B)$ if and only if $A \approx B$. Strong shift equivalence is an answer to the question: 'when are two shift of finite type conjugate?' But, how to decide whether or not there is a strong shift equivalence from A to B is not known so far. Williams has introduced in [**22**] a weaker equivalence relation called shift equivalence and he conjectured the existence of a shift equivalence relation implies the existence of a strong shift equivalence. His conjecture had been an open question for around 20 years, and it turned out to be false in [**11, 12**].

In this section we establish an analogue of the notion of shift equivalence for reversal cases, which will be called a shift-reversal equivalence; and we will show that the existence of a shift-reversal equivalence dose not guarantee the existence of a half strong shift-reversal equivalence. To do this, we first show that the Lind zeta function is a conjugacy invariant of half strong shift-reversal equivalence.

We established the explicit formula for the Lind zeta function for reversal systems in Chapter 1. The Lind zeta function for a reversal system (X, T, R) of order $2m$ is given by

$$\zeta_{(X,T,R)}(t) = \sqrt{\zeta_{(X,T,R^2)}(t^2)} \prod_{\substack{2k-1|m \\ 1\leq 2k-1 \leq m}} \exp\left(\frac{G_{4k-2}(t^{2k-1})}{2k-1}\right),$$

where $\zeta_{(X,T,R^2)}(t)$ is the Lind zeta function for automorphism R^2 of (X, T) of order m, and $G_{4k-2}(t)$ is the generating function of a flip system $(X_{(4k-2)}, T, R^{2k-1})$.

PROPOSITION 2.3.1. *If (X, T, R) is a reversal system of order $2m$, then the Lind zeta functions for (X, T, R) and $(X, T, T \circ R)$ agree.*

Proof. Since $(T \circ R)^2 = R^2$, it suffices to show that $G_{4k-2}(t)$ agrees for each positive odd divisor $2k-1$ of m. Again, it suffices to show that the generating functions of flip systems (X, T, F) and $(X, T, T \circ F)$ agree, since $(T \circ R)^{2k-1} = T \circ R^{2k-1}$. Recall that the generating function $G_{(X,T,F)}$ of a flip system (X, T, F) is given by

$$G_{(X,T,F)}(t) = \sum_{n=1}^{\infty} \Big(p_{2n-1,0}(X,T,F) t^{2n-1} \\ + \frac{p_{2n,0}(X,T,F) + p_{2n,1}(X,T,F)}{2} t^{2n} \Big).$$

It is clear that $p_{n,1}(X, T, F) = p_{n,0}(X, T, T \circ F)$ for all positive integers n; and since

$$T^n(x) = F(x) = x \quad \Leftrightarrow \quad T^n(Tx) = (T \circ (T \circ F))(Tx) = Tx,$$

for all n, it follows that $p_{n,0}(X, T, F) = p_{n,1}(X, T, T \circ F)$. On the other hand, if n is odd, then $p_{n,0}(X, T, F) = p_{n,1}(X, T, F)$ by the proof of (1.12) in the previous chapter. Therefore, we obtain

$$p_{2n-1,0}(X,T,F) = p_{2n-1,0}(X,T,T \circ F),$$

$$p_{2n,0}(X,T,F) = p_{2n,1}(X,T,T \circ F),$$

and

$$p_{2n,1}(X,T,F) = p_{2n,0}(X,T,T \circ F),$$

and hence

$$G_{(X,T,F)}(t) = G_{(X,T,T\circ F)}(t).$$

□

Remark. In general, a reversal system (X, T, R) is not conjugate to $(X, T, T \circ R)$. See [**10**].

Now, suppose that (A, J) and (B, K) are reversal pairs, and that n is a positive integer. A pair (D, E) of matrices over $\mathbb{N}$ satisfying

$$A^n = DE, \; B^n = ED, \; AD = DB, \; EA = BE,$$

$$DK = JE^{\mathsf{T}}, \text{ and } EJ = KD^{\mathsf{T}}$$

is called a *shift-reversal equivalence of lag n from (A, J) to (B, K)*. If there is a shift-reversal equivalence of lag n from (A, J) to (B, K), then we write $(D, E) : (A, J) \sim (B, K)$ (lag n). It is obvious that

$$(D, E) : (A, J) \approx (B, K) \text{ (lag } n) \Rightarrow (D, E) : (A, J) \sim (B, K) \text{ (lag } n).$$

The following example, however, shows that the converse is not true in general.

EXAMPLE 2.3.1. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } \ K = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then (A, J) and (A, K) are reversal pairs, and $J^2 = K^2 = I$. Since $AJ = AK$, it follows that $A^n K = J(A^n)^{\mathsf{T}}$ for all positive integers n; and $(A^n, A^n) : (A, J) \sim (A, K)$ (lag $2n$) is obtained for all n.

The generating functions of shift-flip systems of finite type $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{K,A}})$ are given by

$$G_{(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})}(t) = \frac{2t^2 + 2t^3 + 3t^4 + 2t^5 + 3t^6}{1 - t^4 - 2t^6},$$

and

$$G_{(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{K,A}})}(t) = \frac{t^2 + t^4 + t^6}{1 - t^4 - 2t^6}.$$

Thus, there does not exist a half strong shift-reversal equivalence from (A, J) to (A, K) by Proposition 2.3.1.

We conclude this section with the following question:

QUESTION. Let (A, J) and (B, K) be reversal pairs. Suppose that A and B are irreducible. If the Lind zeta functions of $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$ agree, then is there either a half strong shift-reversal equivalence from (A, J) to (B, K)?

The following example shows that the answer to the question is 'no' when A and B are not irreducible:

EXAMPLE 2.3.2. Let (A, J) and (B, K) be reversal pairs given by

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$ are shift-flip systems of finite type and the generating functions of $(\mathsf{X}_A, \sigma_A, \widetilde{\varphi_{J,A}})$ and $(\mathsf{X}_B, \sigma_B, \widetilde{\varphi_{K,B}})$ are both 0. It is not hard to show that there does not exist a shift-reversal equivalence from (A, J) to (B, K). Thus, there does not exist a half strong shift-reversal equivalence between them, either.

CHAPTER 3

Krieger Presentations for Sofic Shift-Reversal Systems of Finite Order

Suppose that X is a sofic shift, φ is a one-block reversal for (X, σ_X) with a symbol map τ and that φ is of order $2m$ for some positive integer m. We prove Proposition and Proposition by showing that Krieger's joint state chain of X has a natural one-block reversal and the resulting shift-reversal system has the desired properties. We begin with the construction of Krieger's joint state chain.

Let X^+ and X^- denote the set of right-infinite sequences and the set of left-infinite sequences that appear in X, respectively:

$$X^+ = \{x_{[0,\infty)} : x \in X\} \qquad \text{and} \qquad X^- = \{x_{(-\infty,0]} : x \in X\}.$$

If n is a positive integer, $u = u_0 \cdots u_{n-1} \in \mathcal{B}_n(X)$, $\rho \in X^+$ and $\lambda \in X^-$ we define the right-infinite sequence $u\rho$, the left-infinite sequence λu and the bi-infinite sequence $\lambda\rho$ in the usual way:

$$(u\rho)_i = \begin{cases} u_i & 0 \le i \le n-1, \\ \rho_{i-n} & i \ge n, \end{cases}$$

$$(\lambda u)_i = \begin{cases} u_{i+n-1} & -n+1 \le i \le 0, \\ \lambda_{i+n} & i \le -n, \end{cases}$$

and

$$(\lambda\rho)_i = \begin{cases} \rho_i & i \ge 0, \\ \lambda_{i+1} & i \le -1. \end{cases}$$

A subset F of X^+ is said to be a *future* if there is a $\lambda \in X^-$ such that

$$F = \{\rho \in X^+ : \lambda\rho \in X\}$$

and a subset P of X^- is said to be a *past* if there is a $\rho \in X^+$ such that

$$P = \{\lambda \in X^- : \lambda\rho \in X\}.$$

In this case, F and P are called the *future of* λ and the *past of* ρ, respectively. It is well known that there are finitely many futures and pasts ([**9**], [**16**]). If F is a future, P is a past and $a \in \mathcal{B}_1(X)$, we set

$$F(a) = \{\rho \in X^+ : a\rho \in F\} \qquad \text{and} \qquad P(a) = \{\lambda \in X^- : \lambda a \in P\}.$$

It is clear that $F(a)$ is a future whenever $F(a) \neq \varnothing$, and that $P(a)$ is a past whenever $P(a) \neq \varnothing$. A triple (F, a, P) is said to be a *joint state* if F is a future, $a \in \mathcal{B}_1(X)$, P is a past, $F(a) \neq \varnothing$ and $P(a) \neq \varnothing$. Let $\mathcal{A}$ denote the set of all joint states. It is obvious that $|\mathcal{A}| < \infty$. Define the labeling $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ and the zero-one $\mathcal{A} \times \mathcal{A}$ matrix A by

$$\mathcal{L}(F, a, P) = a$$

and

$$A((F_1, a_1, P_1), (F_2, a_2, P_2)) = 1 \;\Leftrightarrow\; F_1(a_1) = F_2 \quad \text{and} \quad P_1 = P_2(a_2).$$

The topological Markov chain X_A is called a *joint state chain* of X. It is well known [**9**] that $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring

Now, we show that the joint state chain of (X, σ_X, φ) has a one-block reversal. Let $\Phi_\tau : X^+ \cup X^- \to X^+ \cup X^-$ be the map defined by

$$\Phi_\tau(\rho_0\rho_1\rho_2\cdots) = \cdots\tau(\rho_2)\tau(\rho_1)\tau(\rho_0) \qquad (\rho_0\rho_1\rho_2\cdots \in X^+)$$

and

$$\Phi_\tau(\cdots\lambda_{-2}\lambda_{-1}\lambda_0) = \tau(\lambda_0)\tau(\lambda_1)\tau(\lambda_2)\cdots \qquad (\cdots\lambda_{-2}\lambda_{-1}\lambda_0 \in X^-).$$

Since $\tau^{2m} = \mathrm{id}_{\mathcal{B}_1(X)}$, we have $\Phi_\tau^{2m} = \mathrm{id}_{X^+\cup X^-}$. It is obvious that a subset F of X^+ is a future if and only if $\Phi_\tau(F)$ is a past and that a subset P of

X^- is a past if and only if $\Phi_\tau(P)$ is a future. We also have

$$\Phi_\tau(F(a)) = \Phi_\tau(F)(\tau(a)) \text{ and } \Phi_\tau(P(a)) = \Phi_\tau(P)(\tau(a)) \ (a \in \mathcal{B}_1(X)). \quad (3.1)$$

Thus, (F, a, P) is a joint state chain if and only if $(\Phi_\tau(P), \tau(a), \Phi_\tau(F))$ is a joint state. (3.1) also implies that

$$\Phi_\tau^k(F(a)) = \Phi_\tau^k(F)(\tau^k(a)) \text{ and } \Phi_\tau^k(P(a)) = \Phi_\tau^k(P)(\tau^k(a)) \ (a \in \mathcal{B}_1(X)). \quad (3.2)$$

We define the zero-one $\mathcal{A} \times \mathcal{A}$ matrix J by

$$J\big((F_1, a_1, P_1), (F_2, a_2, P_2)\big) = 1$$
$$\Leftrightarrow \ \big(\Phi_\tau(P_1), \tau(a_1), \Phi_\tau(F_1)\big) = (F_2, a_2, P_2).$$

It is obvious that $\mathcal{L} \circ \tau_J = \tau \circ \mathcal{L}$ and that

$$AJ = JA^\mathsf{T} \qquad \text{and} \qquad J^{2m} = I.$$

We will call the quadruple $(\mathcal{A}, \mathcal{L}, A, J)$ defined as above *Krieger presentation for* (X, σ_X, φ).

Suppose that X and Y are irreducible. Recall that a block $f \in \mathcal{B}(X)$ is finitary (intrinsically synchronizing) if

$$\forall(\lambda, \rho) \in X^- \times X^+, \quad [\lambda f \in X^- \text{ and } f\rho \in X^+ \ \Rightarrow \ (\lambda f)\rho \in X].$$

Let $\mathcal{A}^0$ be the set of joint states $(F, a, P) \in \mathcal{A}$ such that F is the future of a left-infinite sequence in which a finitary block occurs and P is the past of a right-infinite sequence in which a finitary block occurs; and let $\mathcal{L}^0, A^0, J^0$ denote the restrictions of $\mathcal{L}, A, J$ to $\mathcal{A}^0$, respectively. The topological Markov chain X_{A^0} is called a *joint finitary state chain* of X. It is clear that $(F, a, P) \in \mathcal{A}^0$ if and only if $(\Phi_\tau(P), \tau(a), \Phi_\tau(F)) \in \mathcal{A}^0$ and it is known (p.315 of [**9**]) that $\mathcal{L}^0_\infty : \mathsf{X}_{A^0} \to X$ is a factoring. $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ will be called a *finitary presentation.*

3.1. Proofs of Proposition 1.3.7 and Proposition 1.4.4

Suppose that X is a sofic shift and φ is a one-block reversal for (X, σ_X) with a symbol map τ and that $(\mathcal{A}, \mathcal{L}, A, J)$ is a Krieger presentation for a sofic shift (X, σ_X, φ). It is easy to prove that $\mathcal{L}_\infty : \mathsf{X}_A \to X$ has no graph diamonds.

We assume that $|\varphi| = 2$ and prove (5) of Proposition 1.3.7. Our proof is the adoption from [**13**]. Suppose that $\delta \in \{0, 1\}$, $x \in X$ and $\sigma_X^\delta \circ \varphi(x) = x$. Since $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring, there is a $y \in \mathsf{X}_A$ such that $\mathcal{L}_\infty(y) = x$. If we write

$$y_i = (F_i, a_i, P_i) \qquad (i \in \mathbb{Z}),$$

then $F_i(a_i) = F_{i+1}$, $P_i = P_{i+1}(a_{i+1})$ and $a_i = x_i$ for all $i \in \mathbb{Z}$. Suppose that $\delta = 0$. Then $x_i = \tau(x_{-i})$ for all i, and hence we have

$$\begin{aligned} \Phi_\tau(F_{-i})(x_i) &= \Phi_\tau(F_{-i})(\tau(x_{-i})) \\ &= \Phi_\tau(F_{-i}(x_{-i})) = \Phi_\tau(F_{-i}(a_{-i})) = \Phi_\tau(F_{-i+1}). \end{aligned}$$

for all i. From this, we see that $(F_i, x_i, \Phi_\tau(F_{-i})) \in \mathcal{A}$ for all i and the point $y' \in \mathcal{A}^\mathbb{Z}$ defined by

$$y'_i = (F_i, x_i, \Phi_\tau(F_{-i})) \qquad (i \in \mathbb{Z})$$

is actually an element of X_A. Furthermore, it is obvious that $\mathcal{L}_\infty(y') = x$ and $\widetilde{\varphi_{J,A}}(y') = y'$.

Now, suppose that $\delta = 1$. Then $x_i = \tau(x_{-i-1})$ for all i If we write

$$y_i = (F_i, a_i, P_i) \qquad (i \in \mathbb{Z}),$$

then

$$\begin{aligned} \Phi_\tau(F_{-i-1})(x_i) &= \Phi_\tau(F_{-i-1})(\tau(x_{-i-1})) \\ &= \Phi_\tau(F_{-i-1}(x_{-i-1})) = \Phi_\tau(F_{-i-1}(a_{-i-1})) = \Phi_\tau(F_{-i}) \end{aligned}$$

for all i by the same argument as above. The bi-infinite sequence y' defined by

$$y'_i = (F_i, x_i, \Phi_\tau(F_{-i-1})) \qquad (i \in \mathbb{Z})$$

has the desired properties. This completes the proof of Proposition 1.3.7. $\square$

Now we prove Proposition 1.4.4. Suppose that m is odd and that $|\varphi| = 2m$. If $x \in X$ and $\varphi^k(x) = x$ for some positive odd divisor k of m. Then there are futures F_i such that

$$F_i(x_i) = F_{i+1} \qquad \text{and} \qquad \Phi_\tau^k(F_i) = F_i \qquad (i \in \mathbb{Z}).$$

By (3.2) and the same argument as above, the bi-infinite sequence y defined by

$$y_i = (F_i, x_i, \Phi_\tau^k(F_{-i})) \qquad (i \in \mathbb{Z})$$

is an element of X_A. It is obvious that

$$\mathcal{L}_\infty(y) = x \qquad \text{and} \qquad \widetilde{\varphi_{J,A}}^k(y) = y.$$

Similarly, if $\sigma_X \circ \varphi^k(x) = x$ for some positive odd divisor k of m, then the bi-infinite sequence y defined by

$$y_i = (F_i, x_i, \Phi_\tau^k(F_{-i-1})) \qquad (i \in \mathbb{Z})$$

is an element of X_A and we have

$$\mathcal{L}_\infty(y) = x \qquad \text{and} \qquad \sigma_A \circ \widetilde{\varphi_{J,A}}^k(y) = y.$$

This completes the proof of Proposition 1.4.4. $\square$

Remark. By the same argument as in the above proof, the finitary presentation $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ satisfies the condition (5) of Proposition 1.4.4. It is obvious that $\mathcal{L}^0_\infty$ has no graph diamonds.

3.2. Proof of Proposition 2.2.4

Suppose that φ and ψ are one-block reversals for sofic shifts (X, σ_X) and (Y, σ_Y) with symbol maps τ and μ, respectively. Let $(\mathcal{C}, \mathcal{L}, A, J)$ and $(\mathcal{D}, \mathcal{K}, B, K)$ be Krieger presentations for (X, σ_X, φ) and (Y, σ_Y, ψ), respectively.

Let $\mathcal{C}_f$ be the set of pairs (F, a) such that $F \subset X^+$ is a future, $a \in \mathcal{B}_1(X)$, and $F(a) \neq \varnothing$; and let $\mathcal{C}_p$ be the set of pairs (a, P) such that $a \in \mathcal{B}_1(X)$,

$P \subset X^-$ is the past, and $P(a) \neq \varnothing$:

$$\mathcal{C}_f = \{(F, a) : (F, a, P) \in \mathcal{C} \text{ for some past } P \subset X^-\}$$

and

$$\mathcal{C}_p = \{(a, P) : (F, a, P) \in \mathcal{C} \text{ for some future } F \subset X^+\}.$$

We define $\pi_f : \mathsf{X}_A \to (\mathcal{C}_f)^{\mathbb{Z}}$ and $\pi_p : \mathsf{X}_A \to (\mathcal{C}_p)^{\mathbb{Z}}$ by

$$\pi_f\big((F_i, a_i, P_i)_{i\in\mathbb{Z}}\big) = (F_i, a_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i, P_i)_{i\in\mathbb{Z}} \in \mathsf{X}_A\big)$$

and

$$\pi_p\big((F_i, a_i, P_i)_{i\in\mathbb{Z}}\big) = (a_i, P_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i, P_i)_{i\in\mathbb{Z}} \in \mathsf{X}_A\big);$$

and we write

$$X_f = \pi_f(\mathsf{X}_A) \qquad \text{and} \qquad X_p = \pi_p(\mathsf{X}_A).$$

Let $\mathcal{L}_f : X_f \to X$ and $\mathcal{L}_p : X_p \to X$ be maps defined by

$$\mathcal{L}_f\big((F_i, a_i)_{i\in\mathbb{Z}}\big) = (a_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i)_{i\in\mathbb{Z}} \in X_f\big)$$

and

$$\mathcal{L}_p\big((a_i, P_i)_{i\in\mathbb{Z}}\big) = (a_i)_{i\in\mathbb{Z}} \qquad \big((a_i, P_i)_{i\in\mathbb{Z}} \in X_p\big).$$

Observe that (X_f, σ_{X_f}) and (X_p, σ_{X_p}) are shifts of finite type and that $\mathcal{L}_f : X_f \to X$ and $\mathcal{L}_p : X_p \to X$ are one-block factoring. In a similar way, we define $\mathcal{D}_f$, $\mathcal{D}_p$, $\pi'_f : \mathsf{X}_B \to (\mathcal{D}_f)^{\mathbb{Z}}$, $\pi'_p : \mathsf{X}_B \to (\mathcal{D}_p)^{\mathbb{Z}}$, Y_f, Y_p, $\mathcal{K}_f : Y_f \to Y$, and $\mathcal{K}_p : Y_p \to Y$ as follows:

$$\mathcal{D}_f = \{(F, a) : (F, a, P) \in \mathcal{D} \text{ for some past } P \subset Y^-\},$$

$$\mathcal{D}_p = \{(a, P) : (F, a, P) \in \mathcal{D} \text{ for some future } F \subset Y^+\},$$

$$\pi'_f\big((F_i, a_i, P_i)_{i\in\mathbb{Z}}\big) = (F_i, a_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i, P_i)_{i\in\mathbb{Z}} \in \mathsf{X}_B\big),$$

$$\pi'_p\big((F_i, a_i, P_i)_{i\in\mathbb{Z}}\big) = (a_i, P_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i, P_i)_{i\in\mathbb{Z}} \in \mathsf{X}_B\big),$$

$$Y_f = \pi'_f(\mathsf{X}_B), \qquad Y_p = \pi'_p(\mathsf{X}_B),$$

$$\mathcal{K}_f\big((F_i, a_i)_{i\in\mathbb{Z}}\big) = (a_i)_{i\in\mathbb{Z}} \qquad \big((F_i, a_i)_{i\in\mathbb{Z}} \in Y_f\big)$$

and

$$\mathcal{K}_p\big((a_i, P_i)_{i\in\mathbb{Z}}\big) = (a_i)_{i\in\mathbb{Z}} \qquad \big((a_i, P_i)_{i\in\mathbb{Z}} \in Y_p\big).$$

The following theorems are Krieger's results in [**9**].

THEOREM 3.2.1. *Suppose that* $\eta : (X, \sigma_X) \to (Y, \sigma_Y)$ *is a conjugacy. Then there is a unique conjugacy* $\eta_f : (X_f, \sigma_{X_f}) \to (Y_f, \sigma_{Y_f})$ *such that*

$$\eta \circ \mathcal{L}_f = \mathcal{K}_f \circ \eta_f. \tag{3.3}$$

THEOREM 3.2.2. *Suppose that* $\eta : (X, \sigma_X) \to (Y, \sigma_Y)$ *is a conjugacy. Then there is a unique conjugacy* $\eta_p : (X_p, \sigma_{X_p}) \to (Y_p, \sigma_{Y_p})$ *such that*

$$\eta \circ \mathcal{L}_p = \mathcal{K}_p \circ \eta_p. \tag{3.4}$$

THEOREM 3.2.3. *Suppose that* $\eta : (X, \sigma_X) \to (Y, \sigma_Y)$ *is a conjugacy. Then there is a unique conjugacy* $\theta : (\mathsf{X}_A, \sigma_A) \to (\mathsf{X}_B, \sigma_B)$ *such that*

$$\eta_f \circ \pi_f = \pi'_f \circ \theta, \qquad \eta_p \circ \pi_p = \pi'_p \circ \theta \tag{3.5}$$

and

$$\eta \circ \mathcal{L}_\infty = \mathcal{K}_\infty \circ \theta. \tag{3.6}$$

We will show that the conjugacy θ in Theorem 3.2.3 satisfies

$$\theta \circ \widetilde{\varphi_{J,A}} = \widetilde{\varphi_{K,B}} \circ \theta \tag{3.7}$$

We define $\varphi_f : X_f \to X_p$ and $\varphi_p : X_p \to X_f$ by

$$\varphi_f\big((F_i, a_i)_{i\in\mathbb{Z}}\big)_i = (\tau(a_{-i}), \Phi_\tau(F_{-i})) \qquad \big((F_i, a_i)_{i\in\mathbb{Z}} \in X_f\big)$$

and

$$\varphi_p\big((a_i, P_i)_{i\in\mathbb{Z}}\big)_i = (\Phi_\tau(P_{-i}), \tau(a_{-i})) \qquad \big((a_i, P_i)_{i\in\mathbb{Z}} \in X_p\big).$$

It is clear that

$$\pi_f \circ \widetilde{\varphi_{J,A}} = \varphi_p \circ \pi_p \quad \text{and} \quad \pi_p \circ \widetilde{\varphi_{J,A}} = \varphi_f \circ \pi_f, \tag{3.8}$$

and we have

$$\begin{aligned} \varphi_f \circ \sigma_{X_f} &= \sigma_{X_p}^{-1} \circ \varphi_f, & \varphi_p \circ \sigma_{X_p} &= \sigma_{X_f}^{-1} \circ \varphi_p, \\ \varphi \circ \mathcal{L}_f &= \mathcal{L}_p \circ \varphi_f, & \varphi \circ \mathcal{L}_p &= \mathcal{L}_f \circ \varphi_p. \end{aligned} \tag{3.9}$$

In a similar way, we define $\psi_f : Y_f \to Y_p$ and $\psi_p : Y_p \to Y_f$ by

$$\psi_f\big((F_i, a_i)_{i\in\mathbb{Z}}\big)_i = (\mu(a_{-i}), \Phi_\mu(F_{-i})) \qquad \big((F_i, a_i)_{i\in\mathbb{Z}} \in Y_f\big)$$

and

$$\psi_p\big((a_i, P_i)_{i\in\mathbb{Z}}\big)_i = (\Phi_\mu(P_{-i}), \mu(a_{-i})) \qquad \big((a_i, P_i)_{i\in\mathbb{Z}} \in Y_p\big).$$

From (3.9), we see that $\psi_f^{-1} \circ \eta_p \circ \varphi_f$ and $\psi_p^{-1} \circ \eta_f \circ \varphi_p$ are conjugacies satisfying (3.3) and (3.4), respectively. By Theorem 3.2.1 and Theorem 3.2.2, we obtain

$$\psi_f \circ \eta_f = \eta_p \circ \varphi_f \qquad \text{and} \qquad \psi_p \circ \eta_p = \eta_f \circ \varphi_p \tag{3.10}$$

By (3.5), (3.8), and (3.10), we have

$$\pi'_f \circ \theta \circ \widetilde{\varphi_{J,A}} = \pi'_f \circ \widetilde{\varphi_{K,B}} \circ \theta \quad \text{and} \quad \pi'_p \circ \theta \circ \widetilde{\varphi_{J,A}} = \pi'_p \circ \widetilde{\varphi_{K,B}} \circ \theta$$

and (3.7) follows. This completes the proof of Proposition 2.2.4.

Remark. With the finitary state chain versions of Theorem 3.2.1, Theorem 3.2.2, and Theorem 3.2.3, the same arguments yields (3.7) with respect to the restrictions of θ, η_f, η_p, π_f, π_p, π'_f, and π'_p to their joint finitary state chains.

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Abstract(in Korean)

국문초록

기호 역학계의 유한차 자기동형사상에 대한 린드 제타 함수를 구하고, 이차 역행 기호 역학계의 발생함수를 소개한다. 유한차 역행 기호역학계의 린드 제타 함수는 유한차 자기동형사상에 대한 린드 제타 함수와 이차 역행 기호역학계의 발생함수들로 표현된다. 또한 유한차 역행 소픽 기호 역학계의 린드 제타 함수는 행렬로 나타낼 수 있다.

윌리엄스 분해 정리를 토대로 유한차 역행 소픽 기호 역학계에 분해정리를 건설한다. 이를 위해 절반 기초 동형사상을 소개한다. 모든 유한차 역행 소픽 기호 역학계의 동형 사상은 짝수개의 절반 기초 동형 사상의 합성으로 표현 가능하다.

주요어휘: 역행 함수, 역행 기호 역학계, 린드 제타 함수, 윌리엄스 분해 정리

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